# Comprehensive Examination <br> Quantitative Methods Spring, 2022 

Instruction: This exam consists of three parts. You are required to answer all the questions in all the parts.

## Grading policy:

1. Each part will be graded separately, and there are four possible results in each part: H (honor pass), P (PhD pass), M (master pass), and F (fail).
2. Each part contains a precise grade determining algorithm.
3. The grades from the three parts will be summarized in the descending order, after which the overall grade will be determined using the algorithm summarized in the table below:

| Highest | Middle | Lowest | Overall |
| :--- | :--- | :--- | :--- |
| H | H | H | $\mathbf{H}$ |
| H | H | P | $\mathbf{H}$ |
| H | H | M | $\mathbf{P}$ |
| H | H | F | $\mathbf{M}$ |
| H | P | P | $\mathbf{P}$ |
| H | P | M | $\mathbf{P}$ |
| H | P | F | $\mathbf{M}$ |
| H | M | M | $\mathbf{M}$ |
| H | M | F | $\mathbf{M}$ |
| H | F | F | $\mathbf{F}$ |
| P | P | P | $\mathbf{P}$ |
| P | P | M | $\mathbf{P}$ |
| P | P | F | $\mathbf{M}$ |
| P | M | M | $\mathbf{M}$ |
| P | M | F | $\mathbf{M}$ |
| P | F | F | $\mathbf{F}$ |
| M | M | M | $\mathbf{M}$ |
| M | M | F | F |
| M | F | F | F |
| F | F | F | F |

## Part I-203A

Instructions for Part I: Solve every question. For every question in this part, your answer should be numerical; an abstract formula will not be accepted as an answer. Also, you are required to be very specific about the dimension of the a zero vector or zero matrix. If you intend to write a $1 \times 3$ zero vector, you should write it $\left[\begin{array}{lll}0 & 0 & 0\end{array}\right]$ or $\underset{1 \times 3}{0}$. If you simply write " 0 ", it shall be understood to be a scalar.

Hints for Part $I$ : For some questions, it may be useful to recall that if $Z \sim N(0,1)$, then $P(Z>1.645)=5 \%$ and $P(Z>1.96)=2.5 \%$.

Grading policy for Part I: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let $T$ denote the total number of points.

1. If $T \geq 25$, you will get $H$.
2. If $20 \leq T<25$, you will get P .

3 . If $15 \leq T<20$, you will get M.
4. If $T<15$, you will get F .

1. (3 pts.) No derivation is required for this question; your derivation will not be read anyway. Suppose that $Z \sim N(0,1)$, and let $X_{n}=Z^{2} \cdot 1\left(|Z| \leq \frac{1}{n}\right)$. What is $\lim _{n \rightarrow \infty} E\left[X_{n}\right]$ ? Your answer should be a number.
2. (4 pts.) No derivation is required for this question; your derivation will not be read anyway. Suppose that $X_{1}$ and $X_{2}$ are independent standard normal random variables. Let $Y=X_{1}^{2} X_{2}^{2}$. Let $\varphi\left(X_{2}\right)$ denote the solution to $\min _{\varphi} E\left[\left(Y-\varphi\left(X_{2}\right)\right)^{2}\right]$. What is $E\left[\varphi\left(X_{2}\right)\right]$ ? Your answer should be a number.
3. (6 pts.) No derivation is required for this question; your derivation will not be read anyway. Let $X_{1}, \ldots, X_{n}$ be a random sample of size $n$ from $N(\mu, 1)$, and let $\bar{X}$ denote the sample average.
(a) (3 pts.) Compute

$$
\sup _{\mu<0} \lim _{n \rightarrow \infty} \operatorname{Pr}(\sqrt{n} \bar{X}>0) .
$$

Your answer should be a number.
(b) (3 pts.) Compute

$$
\lim _{n \rightarrow \infty} \sup _{\mu<0} \operatorname{Pr}(\sqrt{n} \bar{X}>0)
$$

Your answer should be a number.
4. (4 pts.) No derivation is required for this question; your derivation will not be read anyway. Let $X_{n}$ denote a sequence of random variables such that the PDF $f_{n}$ of $X_{n}$ is given by

$$
f_{n}(x)=\frac{1}{4} 1(n \leq|x| \leq n+1)+\frac{1}{4} 1(|x| \leq 1) .
$$

(a) (2 pts.) Let $1<B<\infty$ be given. What is $\lim _{n \rightarrow \infty} \operatorname{Pr}\left[\left|X_{n}\right| \leq B\right]$ ? Your answer should be a number.
(b) (2 pts.) This is a true-false question. Is $X_{n}=O_{p}(1)$ ?
5. (4 pts.) No derivation is required for this question; your derivation will not be read anyway. Suppose that $X$ and $Y$ are standard normal random variables, and the correlation between them is $\frac{1}{2}$. What is the correlation between $X^{2}$ and $Y$ ? Your answer should be a number. Hint: We can write $Y=\rho X+\varepsilon$ for some $\varepsilon \sim N\left(0,1-\rho^{2}\right)$, where $\rho=\frac{1}{2}$, and $X$ and $\varepsilon$ are independent of each other.
6. (5 pts.) No derivation is required for this question; your derivation will not be read anyway. Consider $X_{1}, \ldots, X_{4}$ i.i.d. $N\left(\mu, \sigma^{2}\right)$. We assume that $\sigma^{2}$ is known to be equal to 1 . We have $H_{0}: \mu=0$ vs. $H_{1}: \mu>0$. Suppose that $C$ is a critical region such that (1) the test of the form "Reject $H_{0}$ if $\left(X_{1}, \ldots, X_{4}\right) \in C$ " is uniformly most powerful test; and (2) the probability of rejecting $H_{0}$ when $\mu=0$ is $5 \%$. What is the probability of rejecting $H_{0}$ when $\mu=0.8225$. Your answer should be either a concrete number, or $\Phi$ evaluated at a concrete number. Here, $\Phi$ denotes the CDF of $N(0,1)$.
7. (4 pts.) No derivation is required for this question; your derivation will not be read anyway. Assume that $X_{1}, X_{2}, \ldots$ are IID. Assume that $X_{i}=1$ with probability $1 / 2$, and $X_{i}=-1$ with probability $1 / 2$. Let $\bar{X}_{n} \equiv n^{-1} \sum_{i=1}^{n} X_{i}$. Finally, let $g(t) \equiv \sin (t)$.
(a) (1 pt.) What is plim $\bar{X}_{n}$ ? Your answer should be a number.
(b) (1 pt.) What is plim $g\left(\bar{X}_{n}\right)$ ? Your answer should be a number. Hint: $\sin (0)=0$, $\sin (\pi / 2)=1, \sin (\pi)=0, \sin (3 \pi / 2)=-1$
(c) (1 pt.) True-false question. Is $\sqrt{n} \bar{X}_{n}=O_{p}(1)$ ?
(d) (1 pt.) True-false question. Is $\sqrt{n} g\left(\bar{X}_{n}\right)=O_{p}(1)$ ?

## Part II - 203B

Instruction for Part II: Solve every question.
Grading policy for Part II: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let $T$ denote the total number of points.

1. If $T \geq 80$, you will get H .
2. $60 \leq T<80$, you will get P .
3. $45 \leq T<60$, you will get M.
4. If $T<45$, you will get F .

Question 1 ( 50 pts ) Consider a panel data model in which we observe $\left\{Y_{i t}\right\}$ with $Y_{i t} \in \mathbf{R}, 1 \leq i \leq n$, and $1 \leq t \leq 3$ (i.e. three time periods). Further suppose that

$$
\begin{equation*}
Y_{i t}=\left(Y_{i t-1}+Y_{i t-2}\right) \beta_{0}+\varepsilon_{i t} \tag{1}
\end{equation*}
$$

and in addition suppose that $E\left[\varepsilon_{i t} \mid Y_{i t-1}, \ldots, Y_{i 1}\right]=0$ for any $t \geq 2$.
(a) (10 pts) Using the stated assumptions show that $\beta_{0}$ solves the moment condition:

$$
\begin{equation*}
E\left[\left(Y_{i 3}-\left(Y_{i 2}+Y_{i 1}\right) \beta_{0}\right)\left(Y_{i 2}+Y_{i 1}\right)\right]=0 \tag{2}
\end{equation*}
$$

(b) (10 pts) What is the appropriate rank condition for the moment condition in (2)? In other words, state a condition that ensures (2) has a unique solution.
(c) (10 pts) Parts (a) and (b) suggest estimating $\beta_{0}$ by regressing $Y_{i 3}$ on $\left(Y_{i 2}+Y_{i 1}\right)$. Formally derive the asymptotic distribution of such an estimator.
(d) (10 pts) Is OLS necessarily an optimal estimator of $\beta_{0}$ in this model? If you answer yes, please explain why. If you answer no, then propose a more efficient estimator and explain why it is more efficient (though a formal result is not needed).
(e) (10 pts) For this question drop the assumption that $E\left[\varepsilon_{i t} \mid Y_{i t-1}, \ldots, Y_{i 1}\right]=0$. Suppose instead $E\left[\varepsilon_{i t} \varepsilon_{i t-1}\right] \neq 0$ but $E\left[\varepsilon_{i t} \varepsilon_{i j}\right]=0$ for any $j \leq t-2-$ e.g. $\varepsilon_{i t}=$ $u_{i t}+u_{i t-1}$ with $\left\{u_{i t}\right\}$ i.i.d. Is the OLS estimator from part (c) still consistent? If yes, then explain why. If not, then propose a consistent estimator and show its consistency formally stating what assumptions you need. (Hint: You can use the fact that the stated conditions imply $E\left[Y_{i j} \varepsilon_{i t}\right]=0$ for any $j \leq t-2$.)

Question 2 ( 50 pts ) Suppose we observe a sample $\left\{Y_{i}, D_{i}, Z_{i}\right\}_{i=1}^{n}$ with $Y_{i} \in \mathbf{R}, D_{i} \in\{0,1\}, Z_{i} \in \mathbf{R}$,

$$
\begin{equation*}
Y_{i}=\alpha_{0}+D_{i} \beta_{0}+\varepsilon_{i} \quad Z_{i}=Z_{i}^{\star}+\eta_{i} \quad E\left[Z_{i}^{\star} \varepsilon_{i}\right]=E\left[\varepsilon_{i}\right]=0 \tag{3}
\end{equation*}
$$

where the variables $\varepsilon_{i}, Z_{i}^{\star}$, and $\eta_{i}$ are unobserved - i.e. $Z_{i}^{\star}$ is a valid instrument but we instead observe $Z_{i}$ which contains the measurement error $\eta_{i}$. Throughout the problem assume that $\eta_{i}$ is independent of $\left(D_{i}, Z_{i}^{\star}, \varepsilon_{i}\right)$ and that $E\left[\eta_{i}\right]=0$.
(a) (10 pts) Show that under the stated assumptions the parameter $\left(\alpha_{0}, \beta_{0}\right)$ satisfies

$$
\begin{equation*}
E\left[\left(Y-\alpha_{0}-\beta_{0} D\right)\binom{1}{Z}\right]=0 \tag{4}
\end{equation*}
$$

(b) (10 pts) Show that if $\operatorname{Cov}\left\{D, Z^{\star}\right\} \neq 0$, then $\left(\alpha_{0}, \beta_{0}\right)$ are the unique solution to (??) (Hint: Be careful that we assumed $\operatorname{Cov}\left\{D, Z^{\star}\right\} \neq 0$ not that $\operatorname{Cov}\{D, Z\} \neq 0$ ).
(c) (10 pts) Building on parts (a) and (b), derive the asymptotic distribution of the two stage least squares estimator for $\left(\alpha_{0}, \beta_{0}\right)$ that employs $Z$ as an instrument for $D$. Clearly state what additional assumptions you need.
(d) (10 pts) Compute the asymptotic variance of the (unfeasible) two stage least squares estimator for $\left(\alpha_{0}, \beta_{0}\right)$ that employs $Z^{\star}$ as an instrument for $D$. Formally show the asymptotic variance is smaller than the one you obtained in part (c).
(e) (10 pts) Formally show that if $Z^{\star} \in\{0,1\}$ and $Z \in\{0,1\}$, then the assumption that $\eta$ is independent of $Z^{\star}$ cannot be satisfied.

## Part III-203C

Instruction for Part III: Solve every question. The total number of points is 100. Allocate your time wisely!

Let $T$ denote the total number of points.

1. If $T \geq 85$, you will get H .
2. $60 \leq T<85$, you will get P .
3. $45 \leq T<60$, you will get M.
4. If $T<45$, you will get F .

## Question 1 (60 points)

Consider a model specified as

$$
\begin{gathered}
Y_{1}=\left\{\begin{array}{ccc}
1 & \text { if } & X_{0}+g\left(X_{1}\right)-\varepsilon_{1} \geq 0 \\
0 & \text { otherwise }
\end{array}\right. \\
Y_{2}=\left\{\begin{array}{ccc}
1 & \text { if } & s\left(X_{2}\right)-\varepsilon_{2} \geq 0 \\
0 & \text { otherwise }
\end{array}\right.
\end{gathered}
$$

and

$$
Y_{3}=\left\{\begin{array}{ccc}
v\left(X_{3}\right)+\varepsilon_{3} & \text { if } & Y_{2}=1 \\
0 & \text { otherwise }
\end{array}\right.
$$

where the random variables $Y_{1}, Y_{2}, Y_{3}$, and the vectors $X_{0}, X_{1}, X_{2}$, and $X_{2}$ are observed and the random variables $\varepsilon_{1}, \varepsilon_{2}$, and $\varepsilon_{3}$ are unobserved. Let $F_{\left(\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}\right) \mid\left(X_{0}, X_{1}, X_{2}, X_{3}\right)}\left(\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3} \mid x_{0}, x_{1}, x_{2}, x_{3}\right)$ denote the cumulative distribution function of $\left(\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}\right)$, conditional on $\left(X_{0}, X_{1}, X_{2}, X_{3}\right)=$ $\left(x_{0}, x_{1}, x_{2}, x_{3}\right)$. Assume that, when defined, $F_{\left(\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}\right) \mid\left(X_{0}, X_{1}, X_{2}, X_{3}\right)}\left(\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3} \mid x_{0}, x_{1}, x_{2}, x_{3}\right)$ is differentiable and strictly increasing in $\left(\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}\right)$, but otherwise unknown. Assume also that the functions $g, s$, and $v$ are continuous but otherwise unknown, and that the support of $X_{0}$ is $R$, conditional on any values of $\left(X_{1}, X_{2}, X_{3}\right)$. Denote by $f\left(x_{0}, x_{1}, x_{2}, x_{3}\right)$ the probability density of $\left(X_{0}, X_{1}, X_{2}, X_{3}\right)$.
(a; 10) Obtain an expression, in terms of the given unknown functions and distributions, for the probability that $Y_{1}=1$ conditional on $\left(X_{0}, X_{1}\right)=\left(x_{0}, x_{1}\right)$.

For the remaining questions, you can assume that $\left(\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}\right)$ is distributed independently of $\left(X_{0}, X_{1}, X_{2}, X_{3}\right)$.
(b; 10) What features of the unknown functions and distributions can be identified from the distribution of $Y_{1}$ given $\left(X_{0}, X_{1}\right)$ ? Provide proofs.
(c;7) Specify the probability density function of $\left(Y_{1}, Y_{2}\right)$ given $\left(X_{0}, X_{1}, X_{2}\right)=\left(x_{0}, x_{1}, x_{2}\right)$, in terms of the given unknown functions and distributions.
(d; 8) What features of the unknown functions and distributions can be identified from the distribution of $\left(Y_{1}, Y_{2}\right)$ given $\left(X_{0}, X_{1}, X_{2}\right)$ ? Provide proofs.
(e; 7) Obtain an expression, in terms of the given unknown functions and distributions, for

$$
E\left[Y_{3} \mid Y_{2}=1, \quad\left(X_{0}, X_{1}, X_{2}, X_{3}\right)=\left(x_{0}, x_{1}, x_{2}, x_{3}\right)\right]
$$

(f; 8) What features of the unknown functions and distributions can be identified from the distribution of $\left(Y_{2}, Y_{3}\right)$ given $\left(X_{2}, X_{3}\right)$ ? Provide proofs.
( $\mathrm{g} ; 10$ ) Answer (e) and (f) under the assumption that $\left(\varepsilon_{2}, \varepsilon_{3}\right)$ is distributed $N(\mu, \Omega)$ and independently of $\left(X_{0}, X_{1}, X_{2}, X_{3}\right)$, where $\mu=\left(\mu_{2}, \mu_{3}\right)$ and

$$
\Omega=\left[\begin{array}{ll}
\omega_{22} & \omega_{23} \\
\omega_{23} & \omega_{33}
\end{array}\right] .
$$

(Recall that, when $\left(\varepsilon_{2}, \varepsilon_{3}\right)$ is distributed $N(\mu, \Omega)$, the conditional distribution of $\varepsilon_{3}$ given $\varepsilon_{2}$ is $N\left(\mu_{2}+\frac{\omega_{23}}{\omega_{22}}\left(\varepsilon_{2}-\mu_{2}\right), \omega_{33}-\left(\omega_{23}^{2} / \omega_{22}\right)\right)$. Recall also that if $Z \sim N(0,1), E[Z \mid Z>c]=$ $\phi(-c) / \Phi(-c)$, where $\phi$ and $\Phi$ are, respectively, the pdf and cdf of $Z$.)

## Question 2 (40 points)

Consider now a model where

$$
Y_{1}=\left\{\begin{array}{ccc}
1 & \text { if } & \alpha_{1}+X_{0}+\beta_{1} X_{1}-\varepsilon_{1} \geq 0 \\
0 & \text { otherwise }
\end{array}\right.
$$

and

$$
Y_{2}=\left\{\begin{array}{lll}
1 & \text { if } & Y_{1}=0 \text { and } \alpha_{2}+\beta_{2} X_{2}-\varepsilon_{2} \geq 0 \\
2 & \text { if } & Y_{1}=0 \text { and } \alpha_{2}+\beta_{2} X_{2}-\varepsilon_{2}<0 \\
3 & \text { if } & Y_{1}=1 \text { and } \alpha_{3}+\beta_{3} X_{3}-\varepsilon_{3} \geq 0 \\
4 & \text { if } & Y_{1}=1 \text { and } \alpha_{3}+\beta_{3} X_{3}-\varepsilon_{3}<0
\end{array}\right.
$$

Assume that $\left(\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}\right)$ is distributed independently of $\left(X_{0}, X_{1}, X_{2}, X_{3}\right)$ with a $N(\mu, \Omega)$ distribution, where $\mu=\left(\mu_{1}, \mu_{2}, \mu_{3}\right)$ and

$$
\Omega=\left(\begin{array}{ccc}
\omega & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega
\end{array}\right)
$$

are of unknown values. Assume also that $\left(X_{0}, X_{1}, X_{2}, X_{3}\right)$ is continuously distributed with a density that is positive everywhere on $R^{4}$.
(a; 10) Determine what functions of the parameters are identified from the distribution of $\left(Y_{1}, Y_{2}\right)$ given $\left(X_{0}, X_{1}, X_{2}, X_{3}\right)$ and provide proofs for your claims.
(b; 5) Suppose that you are given a random sample $\left\{Y_{1}^{i}, Y_{2}^{i}, X_{0}^{i}, X_{1}^{i}, X_{2}^{i}, X_{3}^{i}\right\}_{i=1}^{N}$ generated from the above model. After re-parameterizing the model, if needed, in terms of parameters that can be identified, describe how to estimate the identified parameters by a Maximum Likelihood method.
(c;15) What are the asymptotic properties of the estimators that you proposed in (b)? Sketch a proof.
(d; 10) Assume that $\mu$ and $\Omega$ are known. Describe how you would use the data in (b) to test the null hypothesis that $\beta_{2}=\beta_{3}$ versus the alternative that $\beta_{2} \neq \beta_{3}$. Provide justifications.

