Comprehensive Examination
Quantitative Methods
Spring, 2020

Instruction: This exam consists of three parts. You are required to answer all the questions in all the parts.

Grading policy:

1. Each part will be graded separately, and there are four possible results in each part: H (honor pass), P (PhD pass), M (master pass), and F (fail).
2. Each part contains a precise grade determining algorithm.
3. The grades from the three parts will be summarized in the descending order, after which the overall grade will be determined using the algorithm summarized in the table below:

Highest	Middle	Lowest	**Overall**
H	H	H	**H**
H	H	P	**H**
H	H	M	**P**
H	H	F	**M**
H	P	P	**P**
H	P	M	**P**
H	P	F	**M**
H	M	M	**M**
H	M	F	**M**
H	F	F	**F**
P	P	P	**P**
P	P	M	**P**
P	P	F	**M**
P	M	M	**M**
P	M	F	**M**
P	F	F	**F**
M	M	M	**M**
M	M	F	**F**
M	F	F	**F**
F	F	F	**F**

Part I - 203A

Instructions for Part I: Solve every question. For every question in this part, your answer should be numerical; an abstract formula will not be accepted as an answer. Also, you are required to be very specific about the dimension of the a zero vector or zero matrix. If you intend to write a 1×3 zero vector, you should write it $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ or $\underset{1\times 3}{0}$. If you simply write "0", it shall be understood to be a scalar.

Grading policy for Part I: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let T denote the total number of points.

1. If $T \geq 25$, you will get H.
2. If $20 \leq T < 25$, you will get P.
3. If $15 \leq T < 20$, you will get M.
4. If $T < 15$, you will get F.

Question 1 (6 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that the joint PDF of random variables X and Y is given by

$$f_{Y,X}(y,x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-1-3x)^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

(a) (3 pts.) Calculate $E[Y|x]$ at $x = 3$. Your answer should be a number.

(b) (3 pts.) Calculate $E[Y^2]$. Your answer should be a number.

Question 2 (6 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that $X \sim N(0, 2^2)$, and let $X_n = X^2 \cdot 1\left(X \leq -\frac{1}{n}\right)$. What is $\lim_{n\to\infty} E[X_n]$?

Question 3 (6 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that

$$\begin{bmatrix} U \\ V \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Let $M(t) = E\left[e^{tX}\right]$ denote the MGF of $X = U^2 + V^2$. Let $h(t) = dM(t)/dt$. Calculate $h(0)$. Your answer should be a number.

Question 4 (6 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that X_n is a discrete random variable such that $P(X_n = 0) = 1 - \frac{1}{n}$ and $P(X_n = n) = P(X_n = -n) = \frac{1}{2n}$. Let $Y \sim N(0, 1)$ be independent of X_n. It can be shown that $(X_n + 1)Y$ converges in distribution to some $N(0, \sigma^2)$. What is σ^2?

Question 5 (6 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Let $X_1, \ldots, X_n$ be a random sample of size n from $N(\mu, 1)$, and let $\overline{X}$ denote the sample average.

(a) (3 pts.) Compute
$$\sup_{\mu<0} \lim_{n\to\infty} \Pr\left(\sqrt{n}\overline{X} > 1.645\right).$$
Your answer should be a number.

(b) (3 pts.) Compute
$$\lim_{n\to\infty} \sup_{\mu<0} \Pr\left(\sqrt{n}\overline{X} > 1.645\right).$$
Your answer should be a number.

Part II - 203B

Instruction for Part II: Solve every question.

Grading policy for Part II: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let T denote the total number of points.

1. If $T \geq 80$, you will get H.
2. $60 \leq T < 80$, you will get P.
3. $45 \leq T < 60$, you will get M.
4. If $T < 45$, you will get F.

Question 1 (50 points) Consider a simple dynamic panel data model in which $\{Y_{it}\}_{t=0}^{T}$ satisfies

$$Y_{it} = \alpha_i + Y_{it-1}\beta_0 + \varepsilon_{it}.$$

Throughout the problem, assume we observe $Y_i \equiv (Y_{i0}, \ldots, Y_{iT})$ for each individual i and that observations $\{Y_i\}_{i=1}^{n}$ are i.i.d. across individuals i.

(a) (10 points) Is it credible in this model to assume that $E[\varepsilon_{it}|\alpha_i, Y_{i0}, \ldots, Y_{iT}] = 0$? Why or why not? Justify your answer.

(b) (10 points) Suppose that $E[\varepsilon_{it}|\alpha_i, Y_{i0}, \ldots, Y_{it}] = 0$ for all t. Show that then

$$E[(\varepsilon_{it} - \varepsilon_{it-1})(Y_{it-1} - Y_{t-2})] = 0.$$

(c) (10 points) Based on your answer to (b), a researcher proposes estimating β_0 by using first differences. His estimator is given by the expression

$$\hat{\beta}_n = \frac{\sum_{i=1}^{n}\sum_{t=2}^{T}(Y_{it} - Y_{it-1})(Y_{it-1} - Y_{it-2})}{\sum_{i=1}^{n}\sum_{t=2}^{T}(Y_{it-1} - Y_{it-2})^2}$$

Maintaining the assumptions of part (b), find the probability limit of $\hat{\beta}_n$. Rigorously establish your result and state what assumptions are needed.

(d) (10 points) Establish the asymptotic distribution of the estimator in part (c).

(e) (10 points) Another researcher decides to instead estimate β_0 by employing a fixed effects estimator. Is his estimator consistent? Justify your answer.

Question 2 (50 points) Consider the following instrumental variables problem

$$Y = \alpha_0 + X\beta_0 + X^2\gamma_0 + \varepsilon$$

where Y and X are both scalars and observable, and ε is not observable and satisfies $E[\varepsilon] = 0$. Also suppose $E[X\varepsilon] \neq 0$ and $E[X^2\varepsilon] \neq 0$. Assume there exists a scalar valued random variable Z satisfying $E[\varepsilon|Z] = 0$. Throughout the problem, assume we observe and i.i.d. sample $\{Y_i, X_i, Z_i\}_{i=1}^n$.

(a) (10 points) Show that any transformation $g(Z)$ of Z is a valid instrument in the sense that it satisfies the exogeneity condition.

(b) (10 points) Suppose we set Z and Z^2 as instruments for X and X^2. What is the rank condition? That is, state an assumption that lets you show that $(\alpha_0, \beta_0, \gamma_0)$ can be identified from the fact that Z and Z^2 satisfy the exogeneity condition.

(c) (15 points) Propose an instrumental variables estimator based on your answer to part (b) and establish its consistency. State whatever assumptions you need.

(d) (15 points) Suppose that Z is binary so that $Z \in \{0, 1\}$. Is the rank condition in part (b) satisfied? Justify your answer.

Part III - 203C

Instruction for Part III: Solve every subquestion.

Grading policy for Part III: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let T denote the total number of points.

1. If $T \geq 80$, you will get H.
2. $60 \leq T < 80$, you will get P.
3. $45 \leq T < 60$, you will get M.
4. If $T < 45$, you will get F.

Question 1 (100 points)

Consider a binary choice model where the utilities of alternatives 0 and 1, given observable characteristics s, x_1 and x_2 and unobservable subutilities ε_1 and ε_2 are, respectively,

$$U_0 = V_0(s, x_0) + \varepsilon_0$$

and

$$U_1 = V_1(s, x_1) + \varepsilon_1$$

A typical individual chooses alternative 1 if $U_1 \geq U_0$. The individual chooses alternative 0 otherwise. Define the observable variable Y by $Y = 1$ if alternative 1 is chosen; $Y = 0$ otherwise. Denote the joint distribution of $(\varepsilon_0, \varepsilon_1)$ conditional on $(S, X_1, X_2) = (s, x_0, x_1)$ by $F_{\varepsilon_0, \varepsilon_1 | (S, X_1, X_2) = (s, x_0, x_1)}$.

(a; 5 points) Obtain an expression for $\Pr[Y = 1 | (S, X_1, X_2) = (s, x_0, x_1)]$ in terms of the functions V_0 and V_1 and the distribution $F_{\varepsilon_0, \varepsilon_1 | (S, X_1, X_2) = (s, x_0, x_1)}$.

(b; 15 points) Suppose that
(i) the support of $(S, X_0, X_1, \varepsilon_0, \varepsilon_1)$ is R^5,
(ii) for unknown parameter values $\alpha_0, \alpha_1, \gamma_0, \gamma_1, \beta_0, \beta_1$,

$$V_0(s, x_0) = \alpha_0 + \gamma_0 \; s + \beta_0 \; x_0$$

$$V_1(s, x_1) = \alpha_1 + \gamma_1 \; s + \beta_1 \; x_1$$

(iii) $(\varepsilon_0, \varepsilon_1)$ is distributed independently of (S, X_0, X_1), and

(iv) the marginal distribution of $(\varepsilon_0, \varepsilon_1)$ is $N(\mu, \Omega)$, where $\mu = (\mu_1, \mu_2)$ and $\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$ are parameters of unknown values.

Determine what functions of the parameters are identified and provide proofs for your claim.

(c; 30 points) Suppose that you are given a random sample $\{Y^i, X_0^i, X_1^i\}_{i=1}^N$ generated from the model in (b). After re-parameterizing the model in terms of parameters that can be identified, describe how to obtain consistent and asymptotically normal estimators for those parameters. Explain is detail.

(d; 20 points) Suppose that the model is instead one where

$$V_0(s, x_0) = \alpha_0 + \beta_0 \; x_0$$
$$V_1(s, x_1) = \alpha_1 + \beta_1 \; x_1$$

and where for any (x_0, x_1), the distribution $F_{\varepsilon_0, \varepsilon_1 | (X_1, X_2) = (x_0, x_1)}$ is such that

$$Median\left[\varepsilon_1 - \varepsilon_0 | (X_0, X_1) = (x_0, x_1)\right] = 0$$

Assume that the support of $(X_0, X_1, \varepsilon_0, \varepsilon_1)$ is R^4.

Analyze the identification of the parameters $\alpha_0, \beta_0, \alpha_1$ and β_1 and of the conditional distribution of $(\varepsilon_1 - \varepsilon_0)$, given (X_0, X_1).

(e; 30 points) Suppose now that U_1 is observed when $U_1 \geq U_0$ and U_0 is observed when $U_1 < U_0$. In other words, assume that the observable variable Y satisfies

$$Y = \begin{cases} U_1 & if \quad U_1 \geq U_0 \\ U_0 & otherwise \end{cases}$$

Assume that (i) the support of (X_1, X_0) is R^2, (ii)

$$U_1 = \alpha_1 + \beta \; x_1 + \varepsilon_1$$
$$U_0 = \alpha_0 + \beta \; x_0 + \varepsilon_0$$

and (iii) $(\varepsilon_0, \varepsilon_1)$ is distributed independently of (X_1, X_0) with a $N(\mu, \Omega)$ distribution.

Suppose that you are given a random sample $\{Y^i, X_0^i, X_1^i\}_{i=1}^N$ generated from this model, and you are interested in estimating β and deriving the asymptotic distribution of your estimator. Explain in detail how you would proceed.