Spring 2005 UCLA Department of Economics
Written Qualifying Examination in Quantitative Methods

Instructions:
Answer **ALL questions** in Parts I, II, and III
Use a separate answer book for each Part.
You have four hours to complete the exam.
Calculators and other electronic devices are not allowed.

1 Part I

1. (10 pt.) Suppose that $Y_n \sim b(n, \pi/n)$. Prove that Y_n converges in distribution to a Poisson distribution with mean equal to π. You may use the fact that the MGF $M(t)$ of Poisson with mean m is equal to $\exp\left[m\left(e^t - 1\right)\right]$.

2. (10 pt.) Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$. Let

$$Y_n = \frac{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^4}{\left(\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2\right)^2} - 3$$

You are asked to characterize the asymptotic distribution of $\sqrt{n}Y_n$. For this purpose, prove the following:

(a) (1 pt.) Let

$$Z_i = \frac{X_i - \mu}{\sigma}.$$

Note that $Z_i \overset{i.i.d.}{\sim} N(0,1)$. Show that

$$Y_n = \frac{\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \overline{Z}\right)^4 - 3\left(\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \overline{Z}\right)^2\right)^2}{\left(\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \overline{Z}\right)^2\right)^2}$$

(b) (1 pt.) Prove that

$$\operatorname*{plim}_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \overline{Z}\right)^2\right)^2 = 1$$

You may use the fact that

$$\left(\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \overline{Z}\right)^2\right)^2 = \left(\frac{1}{n}\sum_{i=1}^{n}Z_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}Z_i\right)^2\right)^2$$

(c) (2 pt.) Prove that

$$\sqrt{n}\left(\begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} Z_i^4 \\ \frac{1}{n}\sum_{i=1}^{n} Z_i^3 \\ \frac{1}{n}\sum_{i=1}^{n} Z_i^2 \\ \frac{1}{n}\sum_{i=1}^{n} Z_i \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) \xrightarrow{d} N\left(0, \begin{bmatrix} 96 & 0 & 12 & 0 \\ 0 & 15 & 0 & 3 \\ 12 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}\right)$$

You may use the fact that

$$\begin{aligned} E\left[Z_i^8\right] &= 105 \\ E\left[Z_i^7\right] &= 0 \\ E\left[Z_i^6\right] &= 15 \\ E\left[Z_i^5\right] &= 0 \\ E\left[Z_i^4\right] &= 3 \\ E\left[Z_i^3\right] &= 0 \\ E\left[Z_i^2\right] &= 1 \end{aligned}$$

You may want to use the fact that, if X is a random vector, then

$$\mathrm{Var}(X) = E\left[XX'\right] - E\left[X\right]E\left[X'\right]$$

(d) (4 pt.) Show that

$$\frac{1}{\sqrt{n}}\left[\sum_{i=1}^{n}\left(Z_i - \overline{Z}\right)^4 - 3\left(\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \overline{Z}\right)^2\right)^2\right] \xrightarrow{d} N\left(0, \omega\right)$$

for some ω. What is ω? You may use the fact that

$$\begin{aligned} & \frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \overline{Z}\right)^4 - 3\left(\frac{1}{n}\sum_{i=1}^{n}\left(Z_i - \overline{Z}\right)^2\right)^2 \\ = \; & \frac{1}{n}\sum_{i=1}^{n} Z_i^4 - 4\overline{Z}\cdot\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^3\right) + 6\overline{Z}^2\cdot\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^2\right) - 4\overline{Z}^3\cdot\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right) + \overline{Z}^4 \\ & -3\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^2\right)^2 + 6\overline{Z}^2\cdot\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^2\right) - 3\overline{Z}^4 \\ = \; & \frac{1}{n}\sum_{i=1}^{n} Z_i^4 - 4\overline{Z}\cdot\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^3\right) + 12\overline{Z}^2\cdot\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^2\right) - 3\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^2\right)^2 - 6\overline{Z}^4 \\ = \; & \frac{1}{n}\sum_{i=1}^{n} Z_i^4 - 4\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)\cdot\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^3\right) + 12\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)^2\cdot\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^2\right) \\ & -3\left(\frac{1}{n}\sum_{i=1}^{n} Z_i^2\right)^2 - 6\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)^4 \end{aligned}$$

(e) (1 pt.) Derive the asymptotic distribution of $\sqrt{n}Y_n$.

2 Part II

1. Consider the classical normal linear regression model

$$Y = X\beta + \varepsilon$$

where $\varepsilon|X \sim N(0, \sigma^2 I_n)$. Describe the Wald test for testing a set of p linear hypotheses on β of the form $H_0 : \Gamma\beta = \gamma_0$ where Γ is a $p \times K$ full row rank matrix. Derive the sampling and asymptotic distribution of the test statistic.

2. Suppose you have n i.i.d. observations on (Y_i, X_i) from the logit model

$$\Pr(Y_i = 1|X_i) = \frac{exp(X_i\beta)}{1 + exp(X_i\beta)}$$

   (a) Describe the MLE of β and show that it is consistent and asymptotically normal. Discuss estimation of its asymptotic covariance matrix.

   (b) Provide (and justify) an alternative estimator that is as efficient asymptotically as the MLE. Show that it is consistent and asymptotically normal.

   (c) Provide an estimator of the marginal effect of a continuous regressor X_{ij} on the conditional probability of "success" and discuss how to construct a 95% confidence interval around your estimate.

Make sure to state all necessary conditions and theorems you use in your proofs.

3 Part III

1. True/Questionable/False? (No points are given for just stating true/questionable or false. The explanation is what counts.)

   (a) In the scalar location model $y_i = \theta + u_i$ with $u_i \sim iid\ N(0, \sigma^2)$ there exists a threshold $\sigma^2_{TH} < \infty$ such that OLS is more efficient than LAD if and only if $\sigma^2 \leq \sigma^2_{TH}$.

   (b) The linear difference equation $y_t = \alpha y_{t-2} + u_t$ where $u_t \sim iid\ N(0, \sigma^2)$ has a causal stationary solution if $|\alpha| < 1$. (Note, y_t is not an AR(1) here.)

   (c) A positive feature of the LM_{CUE} test of the simple full vector parameter hypothesis $H_0 : \theta = \theta_0$ is that its null rejection probabilities are robust to the strength or weakness of the instruments. Therefore, the confidence region $\{\theta \in R^p; LM_{CUE}(\theta) \leq \chi^2_p(1-\alpha)\}$ has approximate coverage probability $1-\alpha$ for θ_0 even if instruments are weak. (Here $\chi^2_p(1-\alpha)$ denotes the $1-\alpha$ quantile of a chi-square distribution with p degrees of freedom.)

   (d) Assume a central limit theorem applies to the stationary zero mean vector series v_t and we have $T^{-1/2}\sum_{t=1}^{T} v_t \to_d N(0, \Omega)$. Then, the estimator $T^{-1}\sum_{t,s=1}^{T} v_s v_t'$ is typically inconsistent for Ω.

   (e) If $X_n = O_p(n^\delta)$ for some $\delta > 0$ it can not be the case that $X_n = o_p(1)$.

2. Take the linear model $y_i = x_i\beta + e_i$, $E(e_i|x_i) = 0$, where x_i and β are scalars.

   (a) Show that $E(e_i x_i) = 0$ and $E(e_i x_i^2) = 0$.

   (b) Is $z_i = (x_i, x_i^2)$ a valid instrumental variable for estimation of β?

   (c) Write down the formula for the 2SLS estimator of β using z_i as an instrument for x_i (the formula of the 2SLS estimator can be written down even if instruments are invalid). Do 2SLS and OLS differ here?

   (d) Find the efficient GMM estimator of β based on the moment condition $E(z_i(y_i - x_i\beta)) = 0$. Does this differ from 2SLS and/or OLS?

   (a) For a covariance stationary process Y_t derive a formula for the linear projection $\widehat{E}(Y_{t+1}|Y_t)$ of Y_{t+1} on a constant and Y_t in terms of $E(Y_t)$ and the covariances of Y_t at lag $k = 0$ and 1. What does this formula boil down to in the case where Y_t is the AR(2) process $Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + u_t$?

   (b) Prove that there cannot be a stationary solution y_t for the scalar difference equation $y_t = y_{t-1} + \varepsilon_t$, where ε_t is white noise with zero mean and variance $\sigma^2 > 0$. You must not assume that $y_0 \equiv 0$ in your proof.