Comprehensive Examination
Quantitative Methods
Fall, 2019

Instruction: This exam consists of three parts. You are required to answer all the questions in all the parts.

Grading policy:

1. Each part will be graded separately, and there are four possible results in each part: H (honor pass), P (PhD pass), M (master pass), and F (fail).

2. Each part contains a precise grade determining algorithm.

3. The grades from the three parts will be summarized in the descending order, after which the overall grade will be determined using the algorithm summarized in the table below:

Highest	Middle	Lowest	**Overall**
H	H	H	**H**
H	H	P	**H**
H	H	M	**P**
H	H	F	**M**
H	P	P	**P**
H	P	M	**P**
H	P	F	**M**
H	M	M	**M**
H	M	F	**M**
H	F	F	**F**
P	P	P	**P**
P	P	M	**P**
P	P	F	**M**
P	M	M	**M**
P	M	F	**M**
P	F	F	**F**
M	M	M	**M**
M	M	F	**F**
M	F	F	**F**
F	F	F	**F**

Part I - 203A

Instructions for Part I: Solve every question. You are required to be very specific about the dimension of the a zero vector or zero matrix. If you intend to write a 1×3 zero vector, you should write it $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ or $\underset{1\times 3}{0}$. If you simply write "0", it shall be understood to be a scalar.

Grading policy for Part I: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let T denote the total number of points.

1. If $T \geq 25$, you will get H.
2. If $20 \leq T < 25$, you will get P.
3. If $15 \leq T < 20$, you will get M.
4. If $T < 15$, you will get F.

Question 1 (5 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that the joint PDF of random variables X and Y is given by

$$f_{Y,X}(y,x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-1-2x)^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

(a) (3 pts.) Calculate $E[Y|\, x]$ at $x = 3$. Your answer should be a number.

(b) (2 pts.) Calculate $E[Y^2]$. Your answer should be a number.

Question 2 (4 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that $X \sim N(0,1)$, and let $X_n = X^2 \cdot 1\left(X \leq -\frac{1}{n}\right)$. What is $\lim_{n\to\infty} E[X_n]$? Your answer should be a number.

Question 3 (4 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that

$$\begin{bmatrix} U \\ V \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

Let $M(t) = E\left[e^{tX}\right]$ denote the MGF of $X = U^2 + V^2$. Let $g(t) = dM(t)/\,dt$ and $h(t) = d^2M(t)/\,dt^2$. Calculate $h(0) - (g(0))^2$. Your answer should be a number. Hint: Some (but not all) of you may find it useful to recall that the MGF of the standard normal distribution is $\exp\left(\frac{t^2}{2}\right)$.

Question 4 (4 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

For simplicity, let $X = (X_1, X_2)'$. What is $\det(E[XX']) - E[\det(XX')]$? Here, $\det(A)$ denotes the determinant of a matrix A. Your answer should be a number. Hint: If A is a 2×2 matrix such that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then $\det(A) = ad - bc$.

Question 5 (4 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right).$$

There is only one possible value of ρ that leads to the equality

$$E\left[(Y - E[Y|X])^2\right] = E\left[(Y - E[Y])^2\right].$$

What is such a value of ρ? Your answer should be a number.

Question 6 (4 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that X_n is a discrete random variable such that $P(X_n = 0) = 1 - \frac{1}{n}$ and $P(X_n = n) = P(X_n = -n) = \frac{1}{2n}$. Let $Y \sim N(0,1)$ be independent of X_n. It can be shown that $(X_n + 1)Y$ converges in distribution to some $N(0, \sigma^2)$. What is

$$\lim_{n\to\infty} \left(E\left[(X_n + 1)^2 Y^2\right] - \sigma^2\right)?$$

An abstract formula would not be accepted as an answer. Note that the question does not ask you to derive the value of σ^2. It only asks you to derive the limit.

Question 7 (5 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that the support of Y is $\{0, 1\}$, and

$$\Pr[Y = 1|X] = \Phi(X'\beta),$$

where Φ denotes the CDF of $N(0,1)$. Assume that $X = (X_1, X_2)'$ is a 2-dimensional random vector with the support $\{(3,1), (3,3)\}$. Let $\beta = (\beta_1, \beta_2)'$. We know that

$$\Pr[Y = 1|(X_1, X_2) = (3,1)] = 0.025.$$
$$\Pr[Y = 1|(X_1, X_2) = (3,3)] = 0.975.$$

(Hint: In the questions below, some of you may find it useful to recall that $\Phi(1.96) = 0.975$.)

(a) (2.5 pts.) What is β_1? Your answer should be a number.

(b) (2.5 pts.) What is β_2? Your answer should be a number.

Part II - 203B

Instruction for Part II : Solve every question.

Grading policy for Part II: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let T denote the total number of points.

1. If $T \geq 80$, you will get H.
2. $60 \leq T < 80$, you will get P.
3. $45 \leq T < 60$, you will get M.
4. If $T < 45$, you will get F.

Question 1 (50 points) Consider a two period panel data model. For each individual i at time period $t \in \{1, 2\}$ we observe an outcome $Y_{it} \in \mathbf{R}$ and covariates $X_{it} \in \mathbf{R}^d$ satisfying

$$Y_{it} = \alpha_i + X_{it}'\beta + \epsilon_{it}$$

where $E[\epsilon_{it}|X_i] = 0$, $X_i = (X_{i1}, X_{i2})$, and $Y_i = (Y_{i1}, Y_{i2})$. Throughout suppose we have an i.i.d. sample $\{Y_i, X_i\}_{i=1}^n$, $E[\epsilon_{it}^2|X_i] = \sigma^2$ and $E[\epsilon_{i1}\epsilon_{i2}|X_i] = 0$.

(a) (10 points) Let $\Delta Y_i = Y_{i2} - Y_{i1}$ and $\Delta X_i = X_{i2} - X_{i1}$. Show that

$$E[(\Delta Y_i - \Delta X_i'\beta)\Delta X_i] = 0.$$

(b) (10 points) Propose an estimator for β that corresponds to the moment condition you derived in part (a). Establish the consistency of the estimator clearly stating what assumptions you require.

(c) (10 points) Suppose X_{it} contains an individual characteristic that does not change through time, such as gender. Are the assumptions you stated in part (b) satisfied?

(d) (10 points) Establish the asymptotic normality of the estimator you proposed in part (a). Clearly state any assumptions you require to establish such a result.

(e) (10 points) Suppose that we are willing to assume that $E[\alpha_i|X_i] = 0$. Propose an estimator for β that is more efficient than that the one you studied in parts (b)-(d). Provide intuition as to why this new estimator is more efficient (though you do not need to establish any formal results).

Question 2 (50 points) Consider a standard treatment effects model in which each individual i has potential outcomes $(Y_i(0), Y_i(1))$, treatment status indicator $D_i \in \{0, 1\}$ (i.e. $D_i = 1$ indicates individual i was treated), and we observe

$$Y_i = Y_i(0) + D_i(Y_i(1) - Y_i(0)).$$

Throughout the problem assume random assignment, so that D_i is independent of $(Y_i(0), Y_i(1))$, and that we observe an i.i.d. sample $\{Y_i, D_i\}_{i=1}^n$.

(a) (10 points) Given the stated assumptions, formally show that

$$\begin{aligned} E[Y] &= E[Y(0)] + P(D = 1)E[Y(1) - Y(0)] \\ E[YD] &= E[Y(1)]P(D = 1) \end{aligned}$$

(b) (10 points) Recall the average treatment effect (ATE) is defined as $\text{ATE} \equiv E[Y(1) - Y(0)]$. Using the stated assumptions and part (a) show that

$$\text{ATE} = \frac{\text{Cov}(Y, D)}{\text{Var}(D)}.$$

(c) (10 points) Propose an estimator for the ATE and establish its asymptotic normality. Clearly state any assumptions you need.

(d) (10 points) Propose a consistent estimator for the asymptotic variance of the ATE estimator you studied in part (c). You don't need to formally show consistency.

(e) (10 points) Suppose we are interested in testing whether the ATE is zero against the alternative that it is not zero at level α. Propose a test statistic and a critical value for this hypothesis testing problem.

Part III - 203C

Instruction for Part III : Solve every question.

Grading policy for Part III: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let T denote the total number of points.

1. If $T \geq 80$, you will get H.
2. $60 \leq T < 80$, you will get P.
3. $45 \leq T < 60$, you will get M.
4. If $T < 45$, you will get F.

Question 1 (20 points) Let $\{u_t\}_{t\in\mathbb{Z}}$ be an i.i.d. process of standard normal random variables. We know that $E[u_1^4] = 3$, $\Pr(u_1 \leq 0) = 0.5$ and $\Pr(u_1 \leq 1) = 0.8413$.

(a) (5 points) Define $X_t = u_t + u_{t-1}$ for any $t \in \mathbb{Z}$. Find the mean and auto-covariance function of $\{X_t\}_{t\in\mathbb{Z}}$.

(b) (5 points) Define $Y_t = u_t u_{t-1}$ for any $t \in \mathbb{Z}$. Find the mean and auto-covariance function of $\{Y_t\}_{t\in\mathbb{Z}}$.

(c) (5 points) For any $t \in \mathbb{Z}$,

$$W_t = \begin{cases} u_t, & t \text{ even} \\ 2^{-1/2}(u_t^2 - 1), & t \text{ odd} \end{cases} .$$

Is $\{W_t\}_{t\in\mathbb{Z}}$ covariance stationary? Justify your answer.

(d) (5 points) Is $\{W_t\}_{t\in\mathbb{Z}}$ defined in (c) strictly stationary? Justify your answer.

Question 2 (30 points) Let $\{X_t\}_{t\in\mathbb{Z}}$ be a time series generated by

$$X_t = u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2}, \text{ where } \{u_t\}_{t\in\mathbb{Z}} \sim WN(0, \sigma^2),$$

where θ_1 and θ_2 are finite real numbers. Let $\gamma_X(h)$ and $\rho_X(h)$ ($h \in \mathbb{Z}$) denote the auto-covariance function and the auto-correlation function of $\{X_t\}_{t\in\mathbb{Z}}$ respectively.

(a) (10 points) What are the largest and smallest possible values for $\rho_X(1)$?

(b) (10 points) What are the largest and smallest possible values for $\rho_X(2)$?

(c) (10 points) Suppose that $\sigma^2 = 1$. Find the values of θ_1 and θ_2 such that $\gamma_X(0) = 2$, $\gamma_X(1) = 0$, $\gamma_X(2) = -1$, and $\gamma_X(h) = 0$ for $|h| > 2$.

Question 3 (50 points) Suppose that $\{X_t\}_t$ and $\{Y_t\}_t$ are covariance stationary processes satisfying

$$\begin{aligned} X_t - \theta_0 X_{t-1} &= u_t, \\ Y_t - \theta_1 Y_{t-1} &= X_t + v_t, \end{aligned}$$

where $|\theta_0| < 1$, $|\theta_1| < 1$ and $\{(u_t, v_t)'\}_{t\in\mathbb{Z}}$ is an i.i.d. process, i.e., $\{(u_t, v_t)'\}_{t\in\mathbb{Z}} \sim i.i.d.(0, \Sigma)$, where

$$\Sigma = \begin{pmatrix} \sigma_u^2 & \sigma_{u,v} \\ \sigma_{u,v} & \sigma_v^2 \end{pmatrix}.$$

(a) (10 points) Show that $\sum_{h=-\infty}^{\infty} |\Gamma_Y(h)| < \infty$, where $\Gamma_Y(h)$ denotes the autocovariance function of $\{Y_t\}_t$.

(b) (15 points) Suppose that $E[u_1^4] < \infty$ and we have data $\{X_t\}_{t=1}^T$. Construct a consistent estimator of the long-run variance of $\{X_t\}_t$. Show the consistency of your estimator.

(c) (25 points) Suppose that $E[u_1^4 + v_1^4] < \infty$ and we have data $\{Y_t\}_{t=1}^T$. Consider the LS estimator $\widehat{\theta}_1 = \sum_{t=2}^T Y_t Y_{t-1} \Big/ \sum_{t=2}^T Y_{t-1}^2$. Is $\widehat{\theta}_1$ a consistent estimator of θ_1? Justify your answer.