Comprehensive Examination
Quantitative Methods
Fall, 2018

Instruction: This exam consists of three parts. You are required to answer all the questions in all the parts.

Grading policy:

1. Each part will be graded separately, and there are four possible results in each part: H (honor pass), P (PhD pass), M (master pass), and F (fail).
2. Each part contains a precise grade determining algorithm.
3. The grades from the three parts will be summarized in the descending order, after which the overall grade will be determined using the algorithm summarized in the table below:

Highest	Middle	Lowest	**Overall**
H	H	H	**H**
H	H	P	**H**
H	H	M	**P**
H	H	F	**M**
H	P	P	**P**
H	P	M	**P**
H	P	F	**M**
H	M	M	**M**
H	M	F	**M**
H	F	F	**F**
P	P	P	**P**
P	P	M	**P**
P	P	F	**M**
P	M	M	**M**
P	M	F	**M**
P	F	F	**F**
M	M	M	**M**
M	M	F	**F**
M	F	F	**F**
F	F	F	**F**

Part I - 203A

Instruction for Part I: Solve every question]

Grading policy for Part I: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality.

Let T denote the total number of points.

1. If $T \geq 95$, you will get H.
2. If $60 \leq T < 95$, you will get P.
3. If $45 \leq T < 60$, you will get M.
4. If $T < 45$, you will get F.

QUESTION 1 (45 points):

A random vector (Y, X) is such that the conditional pdf of Y given $X = x$ is

$$f_{Y|X=x}(y) = \begin{cases} c\, e^{-\,y} & y > x \\ 0 & otherwise \end{cases}$$

and the marginal pdf of X is

$$f_X(x) = \begin{cases} 1 & \theta < x < 2\theta \\ 0 & otherwise \end{cases}$$

where θ is a parameter with value larger than 0.

(a;10) Derive the value of c and the marginal pdf of Y.

(b;5) Derive the conditional cdf of X given $Y = y$.

(c;5) Derive the Moment Generating Function of Y given $X = x$ and the Moment Generating Function of Y, in terms of θ.

(d;5) Derive an expression, in terms of θ, for $\Pr(2\theta < Y + X < 4\theta)$.

Suppose that you are given N i.i.d. observations $\{X^1, ..., X^N\}$ generated from f_X.

(e;10) Provide a consistent and asymptotically normal estimator, $\widehat{\theta}_N$, for θ. Prove that it satisfies the required properties.

(f;10) Obtain an approximate 95% confidence interval for θ^2. Justify all your steps.

QUESTION 2 (55 points):

Consider a situation where the ability of a typical worker is a random variable A distributed $N\left(\mu, \sigma^2\right)$, where μ and σ^2 are parameters. The payment, W, offered to a worker with ability A to perform a task is distributed $N\left(h(A), \omega^2\right)$, where h is a continuous function and ω^2 is a parameter. The worker accepts the offer if $W \geq \varepsilon$, where ε is distributed $N(0,1)$ independently of (W, A). Let $Z=1$ if a worker accepts the offer and $Z=0$ otherwise. If a worker with ability A performs the task, the time, T, it takes him/her to do it is distributed $\exp(\lambda_1)$ if $A>0$ and $\exp(\lambda_2)$ if $A<0$. Let $\widetilde{T}$ denote a given number.

To answer questions (a)-(e), derive expressions in terms of the given functions and parameters.

(a;10) What is the probability that a worker with ability $A=a$ will accept the offer to perform the task? That is, what is the probability that $Z=1$ given $A=a$?

(b;5) What is the probability that a worker with ability $A=a$ is observed performing the task in time $T>\widetilde{T}$?

(c;5) What is the (unconditional) probability that the task will be performed in time $T>\widetilde{T}$?

(d;5) If a worker is known to have performed the task in time $T>\widetilde{T}$, what is the probability that his/her ability is positive?

(e;5) What is the expected offered payment to a typical worker, if his/her ability is unobserved?

(f;5) Are W and T independent? Explain.

(g;10) Is the function h identified from the joint distribution of (Z, A)? If your answer is YES, provide a proof. If your answer is NO, determine what is the most that can be identified about the function h and provide a proof for your answer.

(h;10) Suppose that $h(a)=\beta\ a$. Determine what parameters (or functions of parameters) are identified from the distribution of (T, Z, A). Provide proofs for your answers.

Part II - 203B

Instruction for Part II : Solve every question.

Grading policy for Part II: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality.

Let T denote the total number of points.

1. If $T \geq 95$, you will get H.
2. If $60 \leq T < 95$, you will get P.
3. If $45 \leq T < 60$, you will get M.
4. If $T < 45$, you will get F.

Question 1 (60 pts) Suppose $Y_i \in \mathbf{R}$ and $Z_i^\star \in \mathbf{R}$ are related according to the following linear relationship

$$Y_i = \alpha + Z_i^\star + \varepsilon_i$$

where $E[\varepsilon | Z_i^\star] = 0$. Unfortunately, we do not observe $Z_i^\star$ but instead observe (Z_{i1}, Z_{i2}) satisfying

$$Z_{i1} = Z_i^\star + \eta_{i1} \qquad\qquad Z_{i2} = Z_i^\star + \eta_{i2}.$$

Throughout, assume $E[\eta_{i1}] = E[\eta_{i2}] = 0$, $E[\eta_{i1}^2] = E[\eta_{i2}^2] = \sigma_\eta^2$, $0 < \text{Var}\{Z_i^\star\} < \infty$, (η_{i1}, η_{i2}) are mutually independent and independent of $(Y_i, Z_i^\star, \varepsilon_i)$, and $\{Y_i, Z_{i1}, Z_{i2}\}_{i=1}^n$ is an i.i.d. sample.

(a) (15 points) Suppose we run a linear regression of Y_i on Z_{i1} and compute

$$(\hat{\alpha}_n, \hat{\beta}_n) = \arg\min_{(a,b)} \frac{1}{n} \sum_{i=1}^n (Y_i - a - Z_{i1} b)^2.$$

Find the probability limit of $\hat{\beta}_n$. Formally justify your answer. (Hint: You may find it useful to recall that since we have only one regressor and a constant, $\hat{\beta}_n = Cov(Y, Z_1)/Var(Z_1)$).

(b) (15 points) Show that under the stated assumptions (α_0, β_0) is the unique solution to

$$E\Big[(Y - \alpha - Z_{i1}\beta)\Big(\begin{array}{c} 1 \\ Z_{i2} \end{array}\Big)\Big] = 0.$$

(c) (15 points) Building on part (b) propose a consistent estimator for (α_0, β_0) and formally derive its asymptotic distribution.

(d) (15 points) Suggest an estimator for (α, β) that is more efficient than the one in part (c) without imposing additional assumptions. Provide intuition (no proof needed) as to why it is more efficient. (Hint: Can you think of additional moment restrictions?)

Question 2 (40 pts) Suppose we run an experiment in which we randomly offer unemployed individuals a chance to enroll in a job-retraining program. Let $T_i = 1$ if individual i is offered the program and $T_i = 0$ if she is not offered the program. Let $D_i = 1$ if individual actually enrolls in the program and $D_i = 0$ if she does not enroll in the program. Let $Y_i(1)$ denote the outcome of individual i when they enroll in the program, and $Y_i(0)$ the outcome of individual i when she does not enroll in the program. We do not observe $(Y_i(0), Y_i(1))$ but instead see Y_i which is equal to

$$Y_i \equiv Y_i(0) + D_i(Y_i(1) - Y_i(0))$$

Throughout, assume an individual not offered the program cannot enroll in the program (i.e. $T_i = 0$ implies $D_i = 0$). However, enrollment in the program is not mandatory, so and individual may be offered the program but she may decide not to enroll in it (i.e. $T_i = 1$ but $D_i = 0$).

(a) (10 points) The notion of *noncompliance* refers to the fact that an individual may be offered treatment ($T_i = 1$) but they may decide not to take it ($D_i = 0$). In the context of the job training program, explain why noncompliance may cause a selection problem in the sense that D_i may not be independent of $(Y_i(0), Y_i(1))$.

(b) (10 points) The average outcome on the untreated is defined as $E[Y(0)]$. Show that

$$E[Y|T = 0] = E[Y(0)],$$

and propose a consistent estimator for $E[Y(0)]$. (Hint: Remember that $(Y_i(1), Y_i(0))$ is independent of T_i by random assignment of T_i).

(c) (10 points) Suppose $Y(0)$ is constant for all individuals. Propose a consistent estimator for

$$E[Y(1)|D = 1] - E[Y(0)|D = 1],$$

which is known as the average treatment effect for the treated (ATT).

(d) (10 points) Is the average treatment effect $E[Y(1) - Y(0)]$ identified when there is noncompliance? Justify your answer.

Part III - 203C

Instruction for Part III: Solve every question.

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Let T denote the total number of points.

1. If $T \geq 95$, you will get H.
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3. If $45 \leq T < 60$, you will get M.
4. If $T < 45$, you will get F.

Question 1 (20 points) Suppose that we have one observation from a probability density function $f_\theta(x)$. Find a uniformly most powerful test of size $\alpha = 0.01$ (with explicit critical region) for

$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta = \theta_1$$

when

(a) (10 points) $f_\theta(x) = \begin{cases} 2[\theta x + (1-\theta)(1-x)], & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$, $\theta_0 = \frac{1}{4}$ and $\theta_1 = \frac{1}{2}$;

(b) (10 points) $f_{\theta_0}(x) = \begin{cases} \frac{1}{\exp(x)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ and $f_{\theta_1}(x) = \begin{cases} \frac{x^2}{2\exp(x)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$.

Question 2 (45 points) Suppose that we are interested in estimating the following model

$$Y_t = X_t\theta_1 + W_{1,t}\theta_2 + W_{2,t}\theta_3 + u_t,$$

where θ_1, θ_2 and θ_3 are unknown parameters,

$$E\left[u_t(1, W_{1,t}, W_{2,t})\right] = (0,0,0) \text{ and } E\left[u_t X_t\right] \neq 0.$$

The regressor X_t has the following reduced-form equation

$$X_t = W_{1,t}\beta_1 + W_{2,t}\beta_2 + Z_t\beta_3 + v_t$$

where $\beta_1\beta_2\beta_3 \neq 0$,

$$E\left[v_t(1, W_{1,t}, W_{2,t}, Z_t)\right] = (0,0,0,0) \text{ and } E\left[u_t Z_t\right] = 0.$$

The variance-covariance matrix of $(W_{1,t}, W_{2,t}, Z_t)$ has full rank for any t. We have an i.i.d. sample $\{(Y_t, X_t, W_{1,t}, W_{2,t}, Z_t)\}_{t=1}^n$.

(a) (20 points) Using the moment conditions $E[u_t W_{1,t}] = 0$, $E[u_t W_{2,t}] = 0$ and $E[u_t Z_t] = 0$, construct a GMM estimator of $(\theta_1, \theta_2, \theta_3)$. Derive the asymptotic distribution of the GMM estimator. List the assumptions you needed.

(b) (25 points) Suppose that we know the values of θ_2 and θ_3. Can you use this information to construct a new estimator of θ_1 which has better properties than the GMM estimator constructed in (a)? Justify your answer.

Question 3 (35 points) Consider the following time series model

$$Y_t = \mu t + \rho Y_{t-1} + u_t \text{ for } t > 0,$$

where $|\rho| < 1$, $\{u_t\}$ is $i.i.d.(0, \sigma_u^2)$ with finite 4-th moment and $Y_0 = 0$. Suppose that we have data $\{Y_t\}_{t=1}^T$ from the above model. Consider the LS estimator

$$\widehat{\rho}_T = \frac{\sum_{t=2}^T Y_t Y_{t-1}}{\sum_{t=2}^T Y_{t-1}^2}.$$

Derive the asymptotic distribution of $\widehat{\rho}_T$.