Question 1 - Solution: Write C_i, S_i, T_i for the consumption of coconuts, seafood, time for agent $i = F, R$. Write L_i^x for labor supplied to industry x by agent $i = F, R$. Write p_x for the price of good $x = C, S$; w_i^x for the wage of i in industry x.

(a) $S = L_F^S + L_F^S$; $C = AL_F^C$

(b) Assume an equilibrium in which coconuts are not produced. Then utility functions reduce to $u(S, T) = ST$ so both Friday and Robinson spend half their time gathering seafood and consume $(0, 12, 12)$. Normalize $w_R^S = 1$. Because Friday and Robinson are perfect substitutes in gathering seafood it follows that $w_F^S = 1$. Because the seafood industry makes zero profit it follows that $p_S = 1$.

If coconuts are not produced three things must be true:

- Friday does not want to work in the coconut industry $\Rightarrow w \leq 1$.
- The coconut industry cannot make a profit $\Rightarrow p_C \leq (1/A)w \leq 1/A$.
- Robinson does not demand coconuts $\Rightarrow$ Robinson's marginal utility for coconuts per dollar is at least as great as his marginal utility for seafood per dollar $\Rightarrow p_C \geq (12)(12)/12 = 12$

Putting these together $\Rightarrow A \leq 1/12$.

(c) Assume an equilibrium in which Friday works both in coconuts and in seafood. Normalize so $w_R^S = 1$. Because Friday and Robinson are perfect substitutes in seafood production, $w_F^S = 1$. Because the seafood industry makes zero profit, $p_S = 1$. Because Friday works in both industries, $w_F^S = w_F^C 1$. Because the coconut industry makes zero profit, $p_C = 1/A$.

Because Friday's wage is the price of his time, his problem is

$$\begin{aligned} &\text{Max} S_F T_F \\ &\text{subjectto} \\ &S_F + T_F = 24 \end{aligned}$$

The solution is $S_F = 12, T_F = 12$.

Robinson's problem is

$$\begin{aligned}\text{Max}(1 + C_R)S_RT_R \\ \text{subjectto} \\ (1/A)C_R + S_R + T_R = 24\end{aligned}$$

The solution is $C_R = 8A - (2/3), S_R = 1/(3A) + 8 = T_R$. Hence $L_R^S = 16 - 1/(3A)$ and $S = 20 + 1/3A$.

Utilities are

$$\begin{aligned}u_R &= [8A - (2/3)][(8 + 1/(3A)]^2 \\ u_F &= (12)(12)\end{aligned}$$

Differentiating u_R with respect to A gives

$$\begin{aligned}\frac{du_R}{dA} &= 2[(8 + 1/(3A)][-1(3A^2)][8A - (2/3)] + 8[(8 + 1/(3A)]^2 \\ &= 4/(9A^2) - 8/(3A) + 64\end{aligned}$$

We know from the part (b) that $A \geq 1/12$ so $\frac{du_R}{dA} > 0$. Conclusions:

- Friday's utility is independent of A.
- Robinson's utility is strictly increasing in A.

Explanation

- Friday works in both industries but does not consume coconuts. His wage, his income, and the prices he pays for the goods he consumes (seafood and his own time) are all independent of A; hence his utility is independent of A.
- Robinson's utility depends on his wage, his income, the price of seafood - all of which are independent of A - and on the price of coconuts - which is strictly decreasing in A; hence his utility is strictly increasing in A.

Question 2 - Solution:

(a) NO. $(3,1,0) >> (2,1,0)$ but $(3,1,0) \sim (2,1,0)$.

(b) NO. $(6,5,0) \succeq (4,4,4)$ and $(0,5,6) \succeq (4,4,4)$ but $(3,5,3) \not\succeq (4,4,4)$.

(c) To show that upper sets are closed, suppose $(x_1^n, x_2^n, x_3^n) \to (x_1, x_2, x_3)$ and that $(x_1^n, x_2^n, x_3^n) \succeq (y_1, y_2, y_3)$ for all n. We need to show that $(x_1, x_2, x_3) \succeq (y_1, y_2, y_3)$. Passing to subsequences and re-ordering if necessary we can assume $x_1^n \geq x_2^n \geq x_3^n$ so that x_2^n is the median so $x_2^n \geq \text{median}\{y_1, y_2, y_3\}$. But then $x_1 \geq x_2 \geq x_3$ and x_2 is the median so $x_2 \geq \text{median}\{y_1, y_2, y_3\}$, which is what we needed to show. Hence upper sets are closed.

The proof that lower sets are closed is the same with inequalities reversed.

(d) Suppose the point (x_1, x_2, x_3) is in the demand set with $x_1 \geq x_2 \geq x_3$. (Up to permutations this is the general case.) Then $x_2 =$ median so money spent on x_3 is wasted and money spent on $x_1 - x_2$ is wasted. Hence we must have $x_1 = x_2$ and $x_3 = 0$.

To find the demand correspondence assume $p_1 \geq p_2 \geq p_3$. There are three cases

- $p_1 = p_2 = p_3 \Rightarrow$ the demand set consists of the three points

$$(m/2p_1, m/2p_1, 0), (m/2p_1, 0, m/2p_1), (0, m/2p_1, m/2p_1)$$

- $p_1 = p_2 > p_3 \Rightarrow$ the demand set consists of the two points

$$(m/(p_1 + p_3), 0, m/(p_1 + p_3)), (0, m/(p_2 + p_3), m/(p_2 + p_3))$$

- $p_1 > p_2 \geq p_3 \Rightarrow$ the demand set consists of the single point

$$(0, m/(p_2 + p_3), m/(p_2 + p_3))$$

1. **Price Wars in Boom and Bust Times** (Rotemburg, Saloner AER, 1986)**:** Two firms i, j engage in repeated price competition in a market for a homogeneous good with time-varying demand $q_t = a_t - p_t$. Here, p_t is the lower of two prices $p_{i,t}, p_{j,t}$ in period $t = 1, 2, ...$, and the demand intercept a_t is an i.i.d. random variable, which takes value $\overline{a}$ (a boom) with probability β, and $\underline{a} < \overline{a}$ (a bust) with probability $1-\beta$. Firms observe the realization of a_t before they set prices $p_{i,t}, p_{j,t}$. Firms produce at zero costs, so period profits are $\pi_{i,t} = \begin{cases} p_{i,t}(a_t - p_{i,t}) & \text{when } p_{i,t} < p_{j,t}, \\ \frac{1}{2}p_{i,t}(a_t - p_{i,t}) & \text{when } p_{i,t} = p_{j,t}, \\ 0 & \text{when } p_{i,t} > p_{j,t}. \end{cases}$ Firms discount future profits at rate $\delta < 1$.

   (a) Assume the firms compete only once, rather than repeatedly. What are equilibrium prices and profits, and symmetric collusive prices (i.e. the monopoly price) and profits as a function of demand a_t?

   Answer: (1 point) Equilibrium prices are $p_i = p_j = 0$ and profits are zero. The monopoly price is $a/2$ and per-firm profit is $a^2/8$.

   (b) Returning to the infinitely repeated game, assume that $\beta = 1$, so demand is always high. For what values of δ can the firms use trigger-strategies to charge monopoly prices on path in a SPE? How does your answer change if $\beta = 0$, so demand is always low?

   Answer: (3 points) Per period profits from charging monopoly prices equal $\overline{a}^2/8$. By slightly undercutting the competitor, a firm can grab the entire market for a proft of (almost) $\overline{a}^2/4$, but then lose all future profits. Thus, sticking to the monopoly price is a SPE if $\overline{a}^2/8 \geq (1-\delta)\overline{a}^2/4$, that is if $\delta \geq \underline{\delta} := 1/2$.

   Since all per period payoffs scale in the same way with the demand intercept, the bound on the discount factor $\underline{\delta} = 1/2$ is the same when demand is always low.

   (c) Now assume $0 < \beta < 1$. For what values of δ can the firms use a trigger-strategy to charge monopoly prices on path in a SPE? Is the temptation to deviate from this equilibrium greater in boom or bust periods? (Hint: First, calculate the discounted expected value of charging monopoly prices forever, starting in the next period)

   Answer: (4 points) Discounted expected profits from this strategy, starting in the next period are $\frac{\delta}{8}(\beta\overline{a}^2 + (1-\beta)\underline{a}^2)$; triggering a price war destroys all future value. Slightly undercutting the monopoly price in a boom (resp. bust) period boosts current profits by $(1-\delta)\overline{a}^2/8$ (resp $(1-\delta)\underline{a}^2/8$). Since $\overline{a} > \underline{a}$, the temptation is greater in boom periods. The equilibrium condition is

$$\begin{aligned}
\frac{\delta}{8}(\beta\overline{a}^2 + (1-\beta)\underline{a}^2) &\geq (1-\delta)\overline{a}^2/8 \\
\frac{\beta\overline{a}^2 + (1-\beta)\underline{a}^2}{\overline{a}^2} &\geq \frac{1-\delta}{\delta} \\
\delta &\geq \delta_* := \frac{1}{1+\beta+(1-\beta)\underline{a}^2/\overline{a}^2}
\end{aligned}$$

(d) Is the lower bound on the discount factor in part (c) lower or higher than in part (b)? Give a brief intuition for your answer.

Answer: (2 points) Since $\underline{a}^2/\overline{a}^2 < 1$, $\delta_* > \underline{\delta} = 1/2$, so the lower bound is higher than in part (b). The temptation to deviate in part (c) is as large as in part (b), but the continuation profits are smaller since the boom may not continue, but turn into bust in the future.

2. **Swapping wallets.** There are two wallets: one is "small", and contains 2^n dollars where $n = 0, ..., N-1$ have equal probability $1/N$; the other one is "big", and contains twice as much money as the small wallet; e.g. if the small wallet contains \$4, the large one contains \$8. Initially, these wallets are randomly assigned to two players $i = A, B$,[1] who can then swap their wallets. Each player only observes the money in her own wallet (but does not know whether she has the small or the big wallet). The players simultaneously choose whether they agree to the swap, and trade occurs when both players agree. Players maximize the expected amount of money in the wallet they end up with.

   (a) What are the type spaces Θ_i in this game of incomplete information, and what are their beliefs $\pi(\theta_{-i}|\theta_i)$ over the other player's type?

   Answer: (2 points) A player's type is the amount of money in her wallet $\Theta_i = \{1, ..., 2^N\}$. If

   - $\theta_i = 1$, then i knows that $\theta_j = 2$
   - $\theta_i = 2^N$ then i knows that $\theta_j = 2^{N-1}$
   - $\theta_i = 2, ..., 2^{N-1}$ then i believes that $\theta_j = \frac{1}{2}\theta_i$ or $\theta_j = 2\theta_i$ with equal probability

   (b) Solve for the the unique Bayes-Nash equilibrium in weakly undominated strategies.

   Answer: (3 points)

   - Clearly, it is weakly dominant for the highest type $\theta_i = 2^N$ not to trade.
   - Since $\theta_i = 2^N$ does not trade, the next lower type $\theta_i = 2^{N-1}$ infers that agreeing to trade may only lead to trade if $\theta_j = 2^{N-2}$, in which case it is not profitable. So $\theta_i = 2^{N-1}$ does not trade either.
   - By induction, all types $\theta_i \geq 2$ choose not to trade.
   - In fact, type $\theta_i = 1$ knows that $\theta_j = 2$, and so finds it weakly dominant to agree to trade. But this does not matter for the equilibrium outcome since $\theta_j = 2$ does not agree to trade.

   Now assume that the small wallet contains 2^n dollars for $n = 0, 1, 2, ...$ with probability $(1-b)b^n$ where $b < 1/2$.

   (c) What are player i's beliefs about the money in j's wallet, when his own contains $\theta_i = 2^n$ dollars? (Hint: Use Bayes' rule to compute i's belief π' that his wallet is the small one)

   Answer: (2 points) For $n = 0$, i.e. if $\theta_i = 2^0 = 1$, i knows that she has the small wallet, and so $\theta_j = 2$. For $n > 0$, the probability of $\theta_i = 2^n$ conditional on

[1] More precisely, each player has a 50% chance of receiving either the small or the big wallet, and this event is independent of the amount of money in the small wallet.

the wallet size is $\Pr(\theta_i = 2^n | small) = (1-b)b^n$ and $\Pr(\theta_i = 2^n | big) = (1-b)b^{n-1}$. Since the prior of having the small wallet equals $\pi = 1/2$ her posterior equals $\pi' = \frac{\pi(1-b)b^n}{\pi(1-b)b^n + (1-\pi)(1-b)b^{n-1}} = \frac{b}{1+b} < \frac{1}{2}$. Thus she believes $\theta_j = 2^{n+1}$ with probability $\frac{b}{1+b} < \frac{1}{2}$ and $\theta_j = 2^{n-1}$ with the residual probability $\frac{1}{1+b} > \frac{1}{2}$.

(d) Solve for the unique symmetric Bayes-Nash equilibrium in pure and weakly undominated strategies.

Answer: (3 points) For the lowest type $\theta_i = 2^0 = 1$, agreeing to trade is weakly dominant, as before. It clearly is an equilibrium for no other type to agree to trade. Indeed, this is the only equilibrium. Assume to the contrary that some type $\theta_i = 2^{n-1}$ with $n > 1$ agrees to trade. This can only be profitable if $\theta_i = 2^n$ also agrees to trade. But $\theta_i = 2^n$ believes that $\theta_j = 2^{n-1}$ with probability $\frac{1}{1+b} > \frac{2}{3}$ and $\theta_j = 2^{n+1}$ with probability $\frac{b}{1+b} < \frac{1}{3}$. Thus the expected profits from trading are at most (in the most optimistic case, where $\theta_j = 2^{n+1}$ agrees to trade) $\frac{1}{1+b}(-2^{n-1}) + \frac{b}{1+b}2^n < \frac{2}{3}(-2^{n-1}) + \frac{1}{3}2^n = 0$. This shows that $\theta_i = 2^n$ does not agree to trade, and hence neither does $\theta_i = 2^{n-1}$.

Answer to 5:

(a) For each type in the set of participating types, there must be global incentive compatibility and the equilibrium payoff must satisfy a participation constraint. For each type x that does not participate, the outside payoff must exceed the utility $u(x,o)$ for all outcomes $\{o(\theta)\}_{\theta\in P}=\{(q(\theta),r(\theta))\}_{\theta\in P}$

(b) Suppose types in $\hat{P}=[\underline{\theta},\overline{\theta}]\subset P$ participate. For incentive compatibility,

$$o(\theta)\underset{\theta}{\succsim}o(x),\ \forall\theta,x\in\hat{P}$$

$$u(\theta,o)=u(\theta,q,r)=r-\frac{B(z)}{A(\theta)}$$

Let $\{q(\theta)\}_{\theta\in P}$ be separating BNE actions.

Then

$$U(\theta)\equiv u(\theta,o(\theta))\geq u(\theta,o(x)),\ \forall\theta,x\in\hat{P}$$

Therefore $x=\theta$ solves $\underset{x\in\hat{P}}{Max}\{u(\theta,o(x))\}\equiv U(\theta)$.

$$u(\theta,o(x))=x-\frac{q(x)}{A(\theta)}.$$

(c) Therefore

$$\frac{\partial u}{\partial x}(\theta,o(x))=1-\frac{q'(x)}{A(\theta)}. \qquad (1)$$

This must be zero at $x=\theta$. Therefore

$$\frac{\partial u}{\partial x}(x,o(x))=1-\frac{q'(x)}{A(x)}=0$$

Substituting into (1),

$$\frac{\partial u}{\partial x}(\theta,o(x))=1-\frac{A(x)}{A(\theta)}.$$

Note that the right hand side is positive for $x<\theta$ and negative for $x>\theta$.

Therefore if the FOC holds, all incentive compatibility constraints are satisfied.

(d) Envelope Theorem

$$U'(\theta)=\frac{\partial}{\partial\theta}u(\theta,o(x))\Big|_{x=\theta}=\frac{B(q(\theta)}{A^2(\theta)}A'(\theta)=\frac{A'(\theta)}{A(\theta)}\frac{B(q(\theta)}{A(\theta)}$$

But

$$U(\theta)=r(\theta)-\frac{B(q(\theta))}{A(\theta)}=\theta-\frac{B(q(\theta))}{A(\theta)}$$

Therefore $\frac{B(q(\theta))}{A(\theta)}=\theta-U(\theta)$

$$U'(\theta)=\frac{A'(\theta)}{A(\theta)}(\theta-A(\theta)$$

Rearranging this equation,

$$\frac{d}{d\theta}[A(\theta)U(\theta)]=A(\theta)U'(\theta)+A'(\theta)U'(\theta)=\theta A'(\theta)$$

Integrating both sides of this equation, it follows that $U(\theta)$ is a level set of

$$K(\theta,u)=A(\theta)u-\int_0^\theta xA'(x)dx.$$

There are two possible education technologies, for T_1 the education cost function is $C_1(\theta,z)=\frac{B_1(z)}{\theta}$, and for T_2 it is $C_2(\theta,z)=\frac{B_2(z)}{1+\theta^2}$

(e)- (f)

$$K_1(\theta,u)=\theta u-\int_0^\theta xdx=\theta u-\tfrac{1}{2}\theta^2,\quad K_2(\theta,u)=(1+\theta^2)u-\int_0^\theta x2xdx=(1+\theta^2)u-\tfrac{2}{3}\theta^3$$

Boundary condition for a separating PBE.

The responders know that the minimum value of any type is zero thus type 0 is paid at least zero if he chooses $z=0$. In a separating equilibrium responders correctly believe that his type is zero he will be paid zero. Since $\underline{U}(0)=0$. It follows that $q(0)=r(0)=U(0)=0$. Then the equilibrium payoff function $U(\theta)$ is the level set $K(\theta,u)=0$.

$$K_1(\theta,u)=\theta u-\tfrac{1}{2}\theta^2=0,\quad K_2(\theta,u)=(1+\theta^2)u-\tfrac{2}{3}\theta^3=0.$$

Therefore

$$U_1(\theta)=\tfrac{1}{2}\theta \text{ and } U_2(\theta)=\frac{\frac{2}{3}\theta^3}{1+\theta^2}.$$

Note that participation constraints are satisfied.

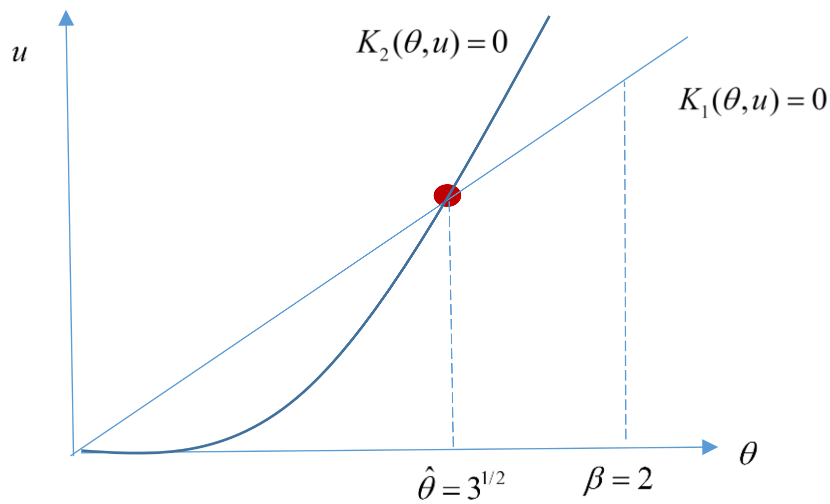


Fig. 5-1 Unique separating PBE for each technology

These are the unique separating PBE payoff functions satisfying the boundary condition.

(g) Note that $\underline{U}(\theta)=U_2(\theta)$ (see part (f)).

Consider the level set $K_1(\theta,U)=0$. We will write this as $U_1(\theta,0)$. Both $U_1(\theta,0)$ and $\underline{U}(0)$. Pass through the origin (see below)

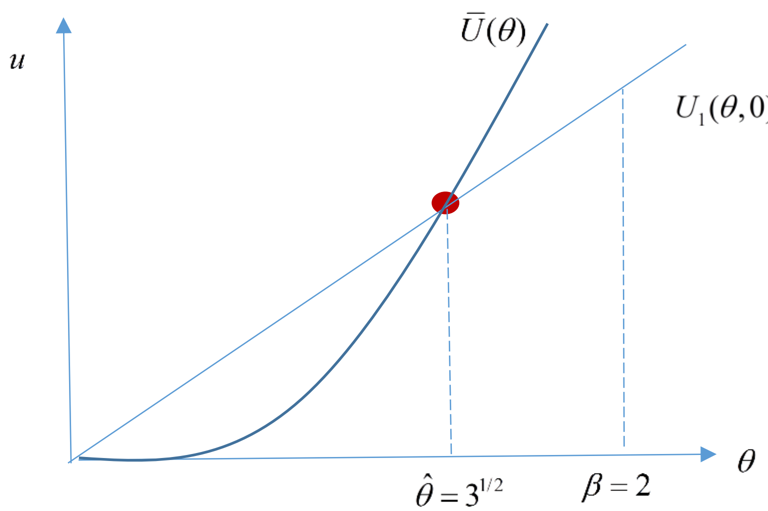


Fig. 5-2 separating PBE

The graph of $U_1(\theta,0)$ has a slope of ½. The graph of $\underline{U}(\theta)$ has a slope of zero at $\theta=0$.

At the point of intersection,

$$\tfrac{2}{3}\frac{\theta^3}{1+\theta^2}=\frac{\theta}{2} \text{ i.e. } \frac{2}{3}\frac{\theta^2}{1+\theta^2}=\frac{1}{2}. \text{ Then } \theta^2=3 \text{ and so } \hat{\theta}=3^{1/2}$$

This is depicted in Fig. 5-2.

Consider another solution to the differential equation $K_1(\theta,u)=\alpha$ where $\alpha<0$. We will write this as $U_1(\theta,\alpha)$. This is depicted in Fig. 5-3

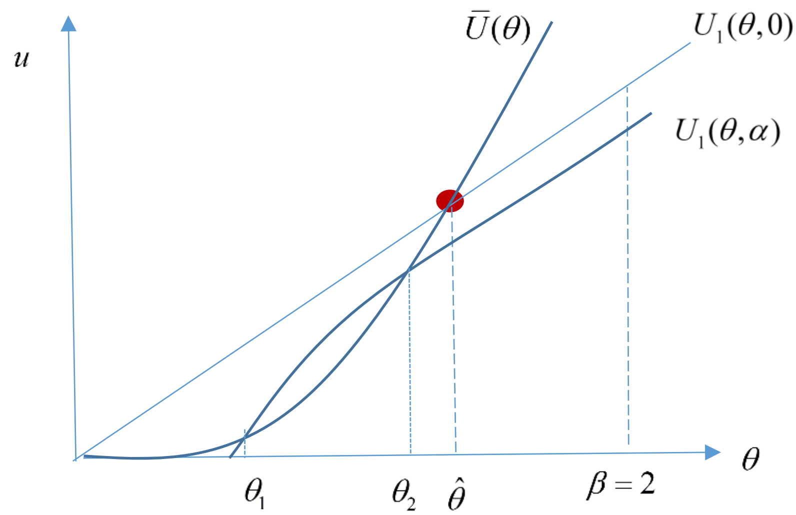


Fig. 5-3 Another separating PBE

The set of participating types is $P(\alpha)=[\theta_1,\theta_2]$.

Note that the boundary types are indifferent between signaling and staking out. This is a PBE because under the belief that a player off the equilibrium path is the worst type, he will be offered a wage of zero. Thus the player will be strictly worse off.

(g) Both signaling technologies available

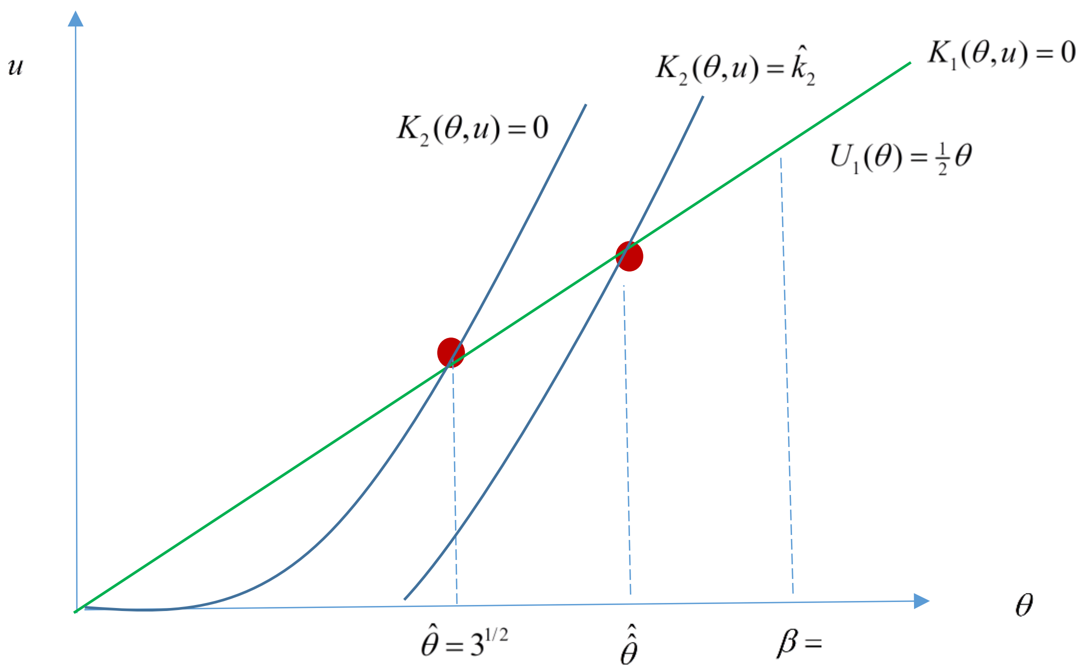


Fig. 5-4 Continuum of PBE

Any level set, $K_2(\theta,u)=\hat{k}_2$, intersecting $U_1(\theta)=\frac{1}{2}\theta$ from below at $\hat{\hat{\theta}}$ is a PBE. All types $\theta>\hat{\hat{\theta}}$ choose technology T_2

Any higher indexed level set yields a higher equilibrium payoff. Therefore the best separating PBE is the one depicted In Fig. 5-5.

Solve for this by finding the type θ^* that solves

$$\underset{\theta}{Max}\{K_2(\theta_1,U_1(\theta))=(1+\theta^2)(\tfrac{1}{2}\theta)-\tfrac{2}{3}\theta^3=\tfrac{1}{2}\theta-\tfrac{1}{6}\theta^3$$

FOC $\frac{1}{2}-\frac{1}{2}\theta^2=1$ so $\theta^*=1$.

The worst separating PBE is the PBE in which all use technology T_1

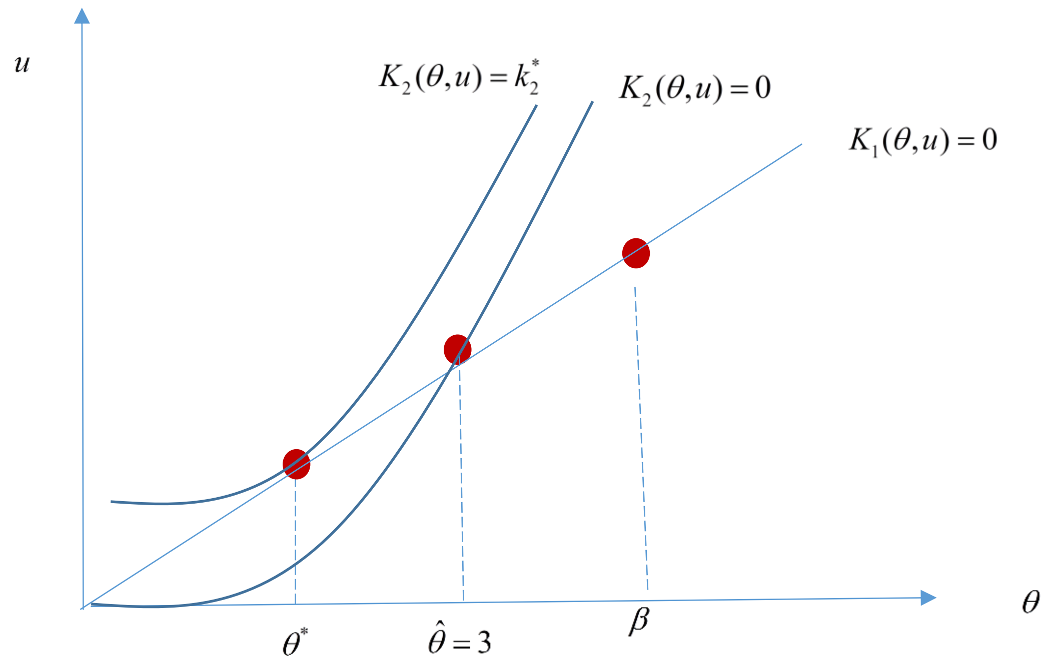


Fig. 5-5 Best separating PBE

Answer to 6.

Incentive compatibility

$$o(x) \underset{\theta}{\precsim} o(\theta),\ \ \forall x \in \Theta$$

If the outcome for type θ is $o \equiv (q,r)$ his payoff is

$$u(\theta,o) = B(\theta,q) - r \qquad (1)$$

The slope of a the level set at (q,r) is

$$\left.\frac{dr}{dq}\right|_{u=\hat{u}} = -\frac{\partial u}{\partial q} / \frac{\partial u}{\partial r} = p(\theta,q)$$

A level set for type θ is depicted in the left diagram below.

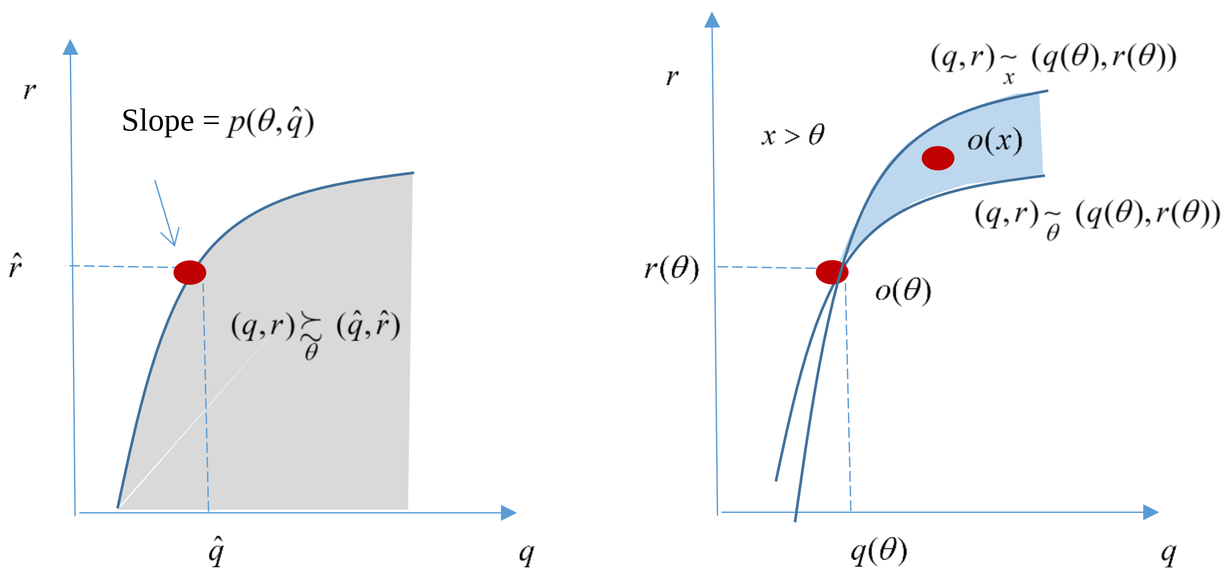


Fig. 6-1: SCP and Monotonicity

The right hand diagram shows the outcome for type θ and the level sets through $o(\theta)$ for type θ and type $x > \theta$. To be incentive compatible $o(x)$ must lie in the sublevel set $u(\theta,q,r) \le u(\theta,q(\theta),r(\theta))$ and the superlevel set $u(x,q,r) \ge u(x,q(\theta),r(\theta))$. This is the shaded region.

Formal proof by contradiction. (NOT REQUIRED)

Suppose that for some θ and $x<\theta$, $q(x)>q(\theta)$.

$q(x)$ is the choice of type x so the extra benefit of choosing $q(x)$ rather than $q(\theta)$ must be less than the extra payoff

$$B(x,q(x))-B(x,q(\theta)=\int_{q(\theta)}^{q(x)} MB(x,q)dq \geq r(x)-r(\theta)$$

Higher types have a higher marginal value of quality, Therefore

$$B(\theta,q(x))-B(\theta,q(\theta)=\int_{q(\theta)}^{q(x)} MB(\theta,q)dq > \int_{q(\theta)}^{q(x)} MB(x,q)dq \geq r(x)-r(\theta)$$

Thus type θ is strictly better off choosing $q(x)$ instead of $q(\theta)$ so $q(\theta)$ is not a best response.

(b) Implications of incentive compatibility

$$U(\theta)\equiv u(\theta,o(\theta))\geq u(\theta,o(x))=B(\theta,q(x))-r(x), \ \forall\theta \text{ and } \forall x\in\Theta \qquad (2)$$

Thus $u(\theta,x)$ takes on its global maximum at $x=\theta$.

Appealing to the Envelope Theorem

$$U'(\theta)=\frac{\partial B}{\partial\theta}(\theta,q)$$

Integrating by parts,

$$\mathbb{E}[U(\theta)]=\int_0^2 U(\theta)F'(\theta)d\theta=U(\theta)(1-F(\theta))\Big|_0^2+\int_0^2 U'(\theta)((1-F(\theta))d\theta$$

$$=U(0)+\mathbb{E}[\frac{U'(\theta)}{h(\theta)}]=U(0)+\mathbb{E}[\frac{\frac{\partial B}{\partial\theta}}{h(\theta)}]$$, where $h(\theta)$ is the hazard rate.

Total surplus $S(\theta,q)$ goes either to the buyers or the designer.

$$\mathbb{E}[u_0(\theta)]=\mathbb{E}[B(\theta,q(\theta))-C(q(\theta))]-\mathbb{E}[U(\theta)]$$

$$=\mathbb{E}[B(\theta,q(\theta))-C(q(\theta))-\frac{\frac{\partial B}{\partial\theta}}{h(\theta)}]-U(0)$$

(c) Define the virtual profit

$$u_0(\theta,q)=B(\theta,q(\theta))-C(q(\theta))-\frac{\frac{\partial B}{\partial\theta}}{h(\theta)}$$

$$=(\tfrac{5}{4}+\theta q)-\tfrac{1}{2}q^2-\frac{1}{h(\theta)}q\,.$$

$$F(x)=\begin{cases}\frac{1}{2}x, & x<1\\ \frac{1}{2}(\frac{x-1+a}{a}), & x\geq 1\end{cases}.$$

$$1-F(x)=\begin{cases}\frac{2-x}{2}, & x<1\\ \frac{2a-x}{2a}, & x\geq 1\end{cases}$$

$$\frac{1}{h(x)}=\frac{1-F(x)}{f(x)}=\begin{cases}2-x, & x\in[0,1)\\ 2a-x, & x\in[1,1+a]\end{cases}$$

$\theta<1$

$$u_0(\theta,q)=(\tfrac{5}{4}+\theta q)-\tfrac{1}{2}q^2-(2-\theta)q$$

$$=(-\tfrac{3}{4}+2\theta)q-\tfrac{1}{2}q^2$$

$\theta\geq 1$

$$u_0(\theta,q)=(\tfrac{5}{4}+\theta q)-\tfrac{1}{2}q^2-(a-\theta)q$$

$$=(\tfrac{5}{4}-2a+2\theta)q-\tfrac{1}{2}q^2$$

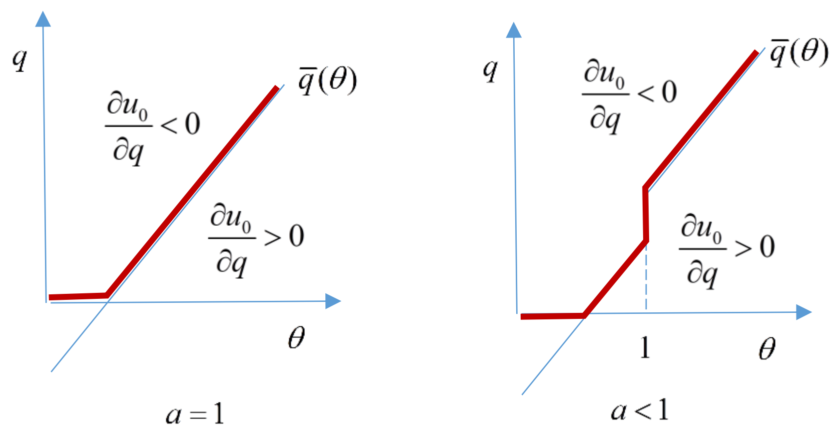
(d) Therefore

$$\frac{\partial u_0}{\partial q}(\theta,q)=-\tfrac{3}{4}+2\theta-q\,,\ \theta\in[0,1) \qquad (3)$$

$$\frac{\partial u_0}{\partial q}(\theta,q)=\tfrac{5}{4}-2a+2\theta-q\,,\ \forall\,\theta\in[1,1+a] \qquad (4)$$

The solution is therefore $q(\theta)=0,\ \forall\theta<\frac{3}{8}$ and $q(\theta)=\frac{1}{2}(\theta-\frac{3}{8}),\ \forall\theta\geq\frac{3}{8}$. Note that, as required, $q(\theta)$ is increasing.

The point-wise maximizer $\bar{q}(\theta)$ is depicted below for the two cases.



Since monotonicity holds, $\bar{q}(\theta)$.

Graphical interpretation. See Fig. 6-2.

$$\bar{q}(\theta)=\begin{cases}2\theta-\frac{3}{4}, & \theta\in[0,1)\\ 2\theta+\frac{5}{4}-2a, & \theta\in[1,1+a]\end{cases}$$

$$\theta(q)=\begin{cases}\frac{3}{8}+\frac{1}{2}q, & q\in[0,\frac{5}{4})\\ \frac{5}{8}-a+\frac{1}{2}q, & q\geq\frac{5}{4}\end{cases}.$$

(e') Graphically, the incentive compatible outcomes implicitly define the mapping

$R(q)$, i.e. $r(\theta)=R(q(\theta))$.

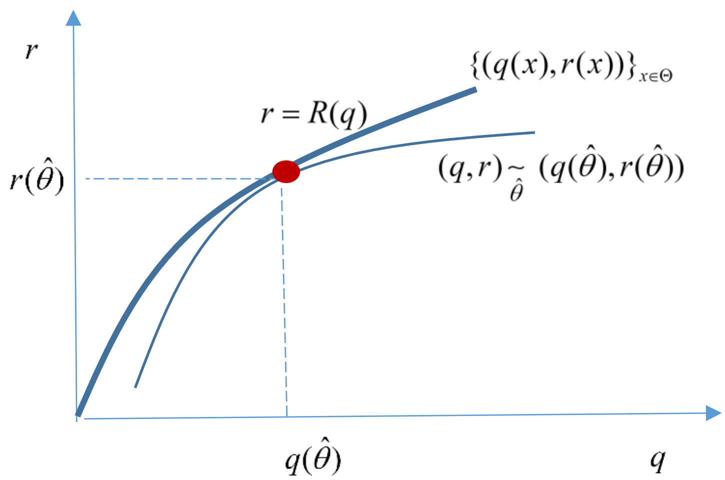


Fig. 6-2 Incentive compatibility

Then the slope of the graph of this function at $q(\theta)$ must be tangential to the level set for type θ at $q(\theta)$.

$$u(\theta,o)=B(\theta,q)-r \ .$$

$$\left.\frac{dr}{dq}\right|_u = -\frac{\partial u}{\partial r}/\frac{\partial u}{\partial q}=\frac{\partial B}{\partial q}=p(\theta,q) \ . \qquad (5)$$

Therefore

$$R'(q(\theta))=p(\theta,q(\theta)) \qquad (6)$$

Substitute into (6) to obtain $R'(q)$. Then integrate with lower boundary condition $(q,R(q))=(0,0)$.

Appealing to (6),

$$R'(q(\theta))=p(\theta,q(\theta)).$$

Then

$$R'(q)=p(\theta(q),q)$$

Where $p(\theta,q)=\frac{5}{4}+\theta-q$

$$\theta(q)=\begin{cases}\frac{3}{8}+\frac{1}{2}q, & q\in[0,\frac{5}{4})\\ \frac{5}{8}-a+\frac{1}{2}q, & q\geq\frac{5}{4}\end{cases}$$

In case (i)

$$p(\theta(q),q)=\frac{5}{4}+\theta(q)-q=\frac{5}{4}+\frac{3}{8}+\frac{1}{2}q-q=2-\frac{1}{2}q.$$

Therefore $R'(q)=2-\frac{1}{2}q$.

Therefore

$$R(q)=2q-\frac{1}{4}q^2+R(0)=2q-\frac{1}{4}q^2$$

In the second case the function R(q) has a gap. There is an intermediate quality range that is not supplied by the monopoly.

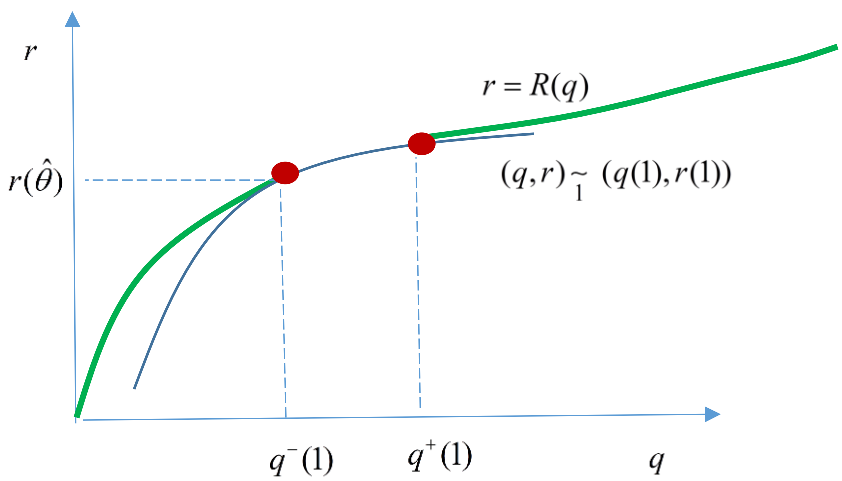
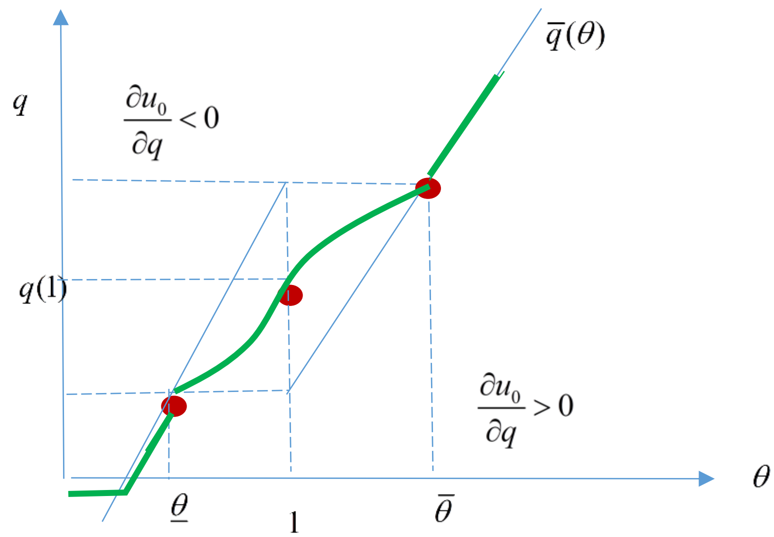


Fig. 6-3 Quality Gap

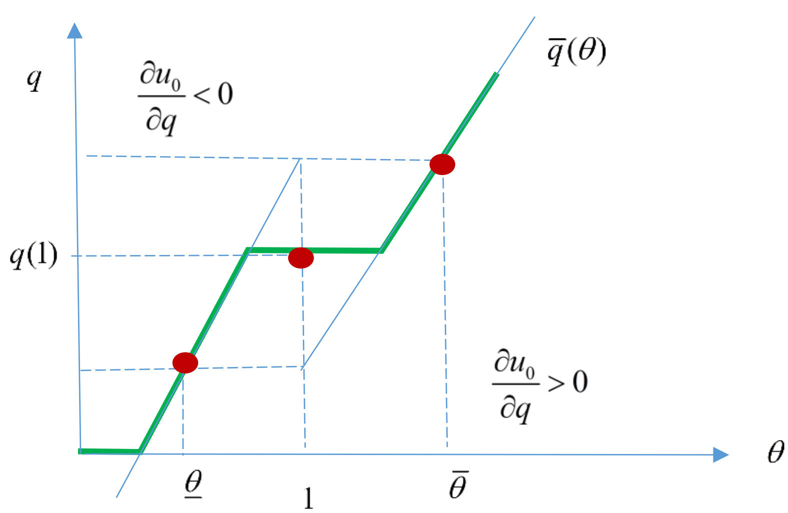
(f)



$a<1$

Pointwise maximization for $\theta < \underline{\theta}$ and $\theta > \bar{\theta}$.

Suppose $q(1)$ is optimal. Consider any monotonic $q(\theta)$ on $[\underline{\theta}, \bar{\theta}]$ as depicted above. This is not optimal since expected profit is pointwise higher below.



$a<1$

Thus there is an interval of types $[\underline{\theta}^*, \bar{\theta}^*]$ who have the same outcome.