Microeconomics Comprehensive Exam: July 3, 2018

Instructions Answer 5 (five) of the following 6 questions. Write each question in a *separate* blue book with your Exam Number on the cover. If you answer more than 5 questions, only the first 5 will be graded.

Question 1 (Robinson Crusoe and Friday)
Robinson and Friday are the only two inhabitants of a small island. They have to subsist on coconuts C and seafood S. Robinson likes both coconuts and seafood, but Friday likes only seafood; both care about their own time):

$$u_R(C_R, S_R, T_R) = (1 + C_R)S_R T_R \ , \ u_F(C_F, S_F, T_F) = S_F T_F$$

(C_R is Robinson's consumption of coconuts, etc.) Both Robinson and Friday can gather seafood; 1 hour gathering seafood (by either of them) produces 1 unit of seafood. However only Friday can climb trees to gather coconuts; 1 hour of climbing trees produces A units of coconuts, where A is a parameter of the economy. Robinson and Friday are each endowed with 24 hours of their own time but no coconuts and no seafood. [Notice that both Robinson and Friday require seafood and time to survive but do not require coconuts.]

(a) Write down the production functions for coconuts and seafood.

The next two parts ask about how the (competitive/Walrasian) equilibrium of the economy depends on the parameter A.

(b) For what range of values (if any) of the parameter A is it the case that (in equilibrium) Friday does *not* gather coconuts and hence coconuts are *not* produced?

(c) Suppose the parameter A is such that (in equilibrium) Friday gathers *both* coconuts *and* seafood.

  (i) Solve for the equilibrium.

  (ii) How do the equilibrium utilities of Friday and Robinson depend on the parameter A? Explain the reason(s) for you answer in words.

Note: You do NOT need to solve for the range of values of the parameter A for which such an equilibrium exists.

Question 2 (Median Preferences)
Define a preference relation $\succeq$ on $\mathbb{R}^3_+$ by by

$$(x_1, x_2, x_3) \succeq (x_1', x_2', x_3') \Leftrightarrow \text{median}\{x_1, x_2, x_3\} \geq \text{median}\{x_1', x_2', x_3'\}$$

This preference relation is complete, transitive and weakly monotone.

(a) Is this preference relation strictly monotone? If so, prove it; if not give a counter-example.

(b) Is this preference relation convex? If so, prove it; if not give a counter-example.

(c) Using the definition of continuity of preferences, show that this preference relation is continuous.

(d) Find the (Marshallian) demand correspondence for this preference relation, as a function of prices p_1, p_2, p_3 and income m. To make life simpler you can assume $p_1 \geq p_2 \geq p_3 > 0$.

Question 3 (Price Wars in Boom and Bust Times)
Two firms i, j engage in repeated price competition in a market for a homogeneous good with time-varying demand $q_t = a_t - p_t$. Here, p_t is the lower of two prices $p_{i,t}, p_{j,t}$ in period $t = 1, 2, ...$, and the demand intercept a_t is an i.i.d. random variable, which takes value $\overline{a}$ (a boom) with probability β, and $\underline{a} < \overline{a}$ (a bust) with probability $1 - \beta$. Firms observe the realization of a_t before they set prices $p_{i,t}, p_{j,t}$. Firms produce at zero costs, so period profits are

$$\pi_{i,t} = \begin{cases} p_{i,t}(a_t - p_{i,t}) & \text{when } p_{i,t} < p_{j,t}, \\ \frac{1}{2} p_{i,t}(a_t - p_{i,t}) & \text{when } p_{i,t} = p_{j,t}, \\ 0 & \text{when } p_{i,t} > p_{j,t}. \end{cases}$$

Firms discount future profits at rate $\delta < 1$.

(a) Assume the firms compete only once, rather than repeatedly. What are equilibrium prices and profits, and symmetric collusive prices (i.e. the monopoly price) and profits as a function of demand a_t?

(b) Returning to the infinitely repeated game, assume that $\beta = 1$, so demand is always high. For what values of δ can the firms use trigger-strategies to charge monopoly prices on path in a SPE? How does your answer change if $\beta = 0$, so demand is always low?

(c) Now assume $0 < \beta < 1$. For what values of δ can the firms use a trigger-strategy to charge monopoly prices on path in a SPE? Is the temptation to deviate from this equilibrium greater in boom or bust periods? (Hint: First, calculate the discounted expected value of charging monopoly prices forever, starting in the next period)

(d) Is the lower bound on the discount factor in part (c) lower or higher than in part (b)? Give a brief intuition for your answer.

Question 4 (Swapping wallets)
There are two wallets: one is "small", and contains 2^n dollars where $n = 0, ..., N-1$ have equal probability $1/N$; the other one is "big", and contains twice as much money as the small wallet; e.g. if the small wallet contains \$4, the large one contains \$8. Initially, these wallets are randomly assigned to two players $i = A, B$ (i.e., each player has a 50% chance of receiving either the small or the big wallet, and this event is independent of the amount of money in the small wallet) who can then swap their wallets. Each player only observes the money in her own wallet (but does not know whether she has the small or the big wallet). The players simultaneously choose whether they agree to the swap, and trade occurs when both players agree. Players maximize the expected amount of money in the wallet they end up with.

(a) What are the type spaces Θ_i in this game of incomplete information, and what are their beliefs $\pi(\theta_{-i}|\theta_i)$ over the other player's type?

(b) Solve for the the unique Bayes-Nash equilibrium in weakly undominated strategies.

Now assume that the small wallet contains 2^n dollars for $n = 0, 1, 2, ...$ with probability $(1-b)b^n$ where $b < 1/2$.

(c) What are player i's beliefs about the money in j's wallet, when his own contains $\theta_i = 2^n$ dollars? (Hint: Use Bayes' rule to compute i's belief π' that his wallet is the small one)

(d) Solve for the unique symmetric Bayes-Nash equilibrium in pure and weakly undominated strategies.

Question 5 (Signaling with continuous types)

The canonical signaling game is a two stage game in which player 0 (the first mover) alone knows her type $\theta \in \Theta$. Player 0 first chooses whether or not to participate. Let P be the set of participating types. If $\theta \in \Theta \setminus P$ the game ends and player 0 has a payoff of $\underline{U}(\theta)$. If $\theta \in P$, player 0 chooses a public action $z \in Z$. In stage 2, player 0 auctions a product or service of value $m(\theta, z)$ to one of the stage players (responders). Let $\tilde{\theta}(z)$ be the updated beliefs of each responder, given the observed action, z. Responders submit sealed bids. The equilibrium highest bid is then a bid of $r = \underset{\tilde{\theta}(z)}{\mathrm{E}}[m(\theta, z)]$

Separating PBE

Let P be a PBE set of signaling types. Let $\{q(\theta)\}_{\theta \in P}$ be the PBE strategy of these types. Let Q be the image of the mapping $q(\cdot)$. In a separating PBE, each type chooses a different signal so there is an inverse mapping $\phi : Q \to P$. Thus the equilibrium response in the second stage of the game is

$$r(q) = m(\phi(q), q)$$

Consider the simple Spence signaling model in which the value (i.e. the marginal product) of a type $\theta \in \Theta = [0,2]$ worker with education z is $m(\theta, z) = \theta$ (unproductive signal). The cost of education z is $C(\theta, z) = B(z)/A(\theta)$ where $A(\cdot)$ and $B(\cdot)$ are positive, strictly increasing, differentiable functions.

(a) Provide a complete description of the constraints that must be satisfied if P is a separating PBE set of signaling types.

(b) Define $\{o(\theta)\}_{\theta \in P} = \{q(\theta), r(\theta)\}_{\theta \in P} = \{q(\theta), \theta\}_{\theta \in P}$. These are the separating PBE outcomes. For each $\theta \in P$, let $U(\theta)$ be the PBE payoff. Explain why $x = \theta$ solves the following maximization problem for all $\theta \in P$

$$\underset{x \in P}{Max}\{\bar{u}(\theta, x) = x - \frac{B(q(x))}{A(\theta)}\}$$

(c) Show that if the necessary conditions hold and $q(\theta)$ is increasing, then all the incentive compatibility constraints are satisfied.

(d) Suppose $\bar{P} = [\underline{\theta}, \bar{\theta}] \subset P$. Show that if $\{o(\theta)\}_{\theta \in \bar{P}}$ is separating, then

$$U'(\theta) = \frac{A'(\theta)}{A(\theta)}(\theta - U(\theta)) \ \ \forall\theta \in \overline{P}$$

Hence show that $U(\theta)$ is a level set of

$$K(\theta, u) = A(\theta)u - \int_0^\theta xA'(x)dx\,.$$

Now suppose that there are two possible education technologies, for T_1 the education cost function is $C_1(\theta, z) = B_1(z)/\theta$, and for T_2 it is $C_2(\theta, z) = B_2(z)/(1+\theta^2)$.

(e) If only education technology T_1 is available and $\underline{U}(\theta) = \frac{1}{4}\theta,\ \forall\theta \in \Theta$, show that there is a unique separating PBE.

(f) If only education technology T_2 is available and $\underline{U}(\theta) = 0,\ \forall\theta \in \Theta$, again characterize any separating PBE.

(g) If only education technology T_1 is available and $\underline{U}(\theta) = \dfrac{2\theta^3}{3(1+\theta^2)}$, show that there is a PBE in which $P = [0, 3^{1/2}]$. Is this unique?

(h) Suppose finally that both educational technologies are available and $\underline{U}(\theta) = 0$. Fully characterize the set of separating PBE.

Question 6 (Monopoly and product quality)

A type $\theta \in \Theta = [0, 1+a]$ customer places a value $B(\theta, q) = (\frac{5}{4} + \theta)q$ on a unit of a product of quality q . The cost per unit of a product of quality q is $C(q) = \frac{1}{2}q^2$. The parameter θ is distributed with the following c.d.f.

$$F(\theta) = \begin{cases} \dfrac{1}{2}\theta, & \theta < 1 \\ \dfrac{1}{2}(\dfrac{\theta - 1 + a}{a}), & \theta \geq 1 \end{cases}.$$

Then the density function is well defined but has a discontinuity at $\theta = 1$, unless $a = 1$

(a) Explain why incentive compatible outcomes $\{o(\theta)\}_{\theta \in \Theta} = \{(q(\theta), r(\theta)\}_{\theta \in \Theta}$ are increasing

(b) Solve for the derivative of $U(\theta) = u(\theta, q(\theta), r(\theta)) = B(\theta, q(\theta)) - r(\theta)$ and hence obtain an expression for $\mathrm{E}[U(\theta)]$

(c) Obtain an expression for the expected profit of the monopoly

$$\mathrm{E}[u_0(\theta)] = \mathrm{E}[B(\theta, q(\theta)) - C(q(\theta))] - \mathrm{E}[U(\theta)]$$

(d) Solve for the profit-maximizing allocation if (i) $a = 1$ (ii) $a < 1$

~~(e) In each case, how much would the monopoly charge for a unit of quality q.~~

~~(f) In each case, how might the profit-maximizing outcomes be implemented?~~

(g) Comment briefly on the profit-maximizing allocation if $a > 1$

(e') In each case how might the profit maximizing outcomes be implemented, i.e. how much would the monopoly charge for a unit of quality q?