Comprehensive Examination
Quantitative Methods
Fall, 2017

Instruction: This exam consists of three parts. You are required to answer all the questions in all the parts.

Grading policy:

1. Each part will be graded separately, and there are four possible results in each part: H (honor pass), P (PhD pass), M (master pass), and F (fail).
2. Each part contains a precise grade determining algorithm.
3. The grades from the three parts will be summarized in the descending order, after which the overall grade will be determined using the algorithm summarized in the table below:

Highest	Middle	Lowest	**Overall**
H	H	H	**H**
H	H	P	**H**
H	H	M	**P**
H	H	F	**M**
H	P	P	**P**
H	P	M	**P**
H	P	F	**M**
H	M	M	**M**
H	M	F	**M**
H	F	F	**F**
P	P	P	**P**
P	P	M	**P**
P	P	F	**M**
P	M	M	**M**
P	M	F	**M**
P	F	F	**F**
M	M	M	**M**
M	M	F	**F**
M	F	F	**F**
F	F	F	**F**

Part I - 203A

Instruction for Part I: Solve every sub-question.
Grading policy for Part I: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let T denote the total number of points.

1. If $T \geq 25$, you will get H.
2. If $20 \leq T < 25$, you will get P.
3. If $15 \leq T < 20$, you will get M.
4. If $T < 15$, you will get F.

QUESTION: Consider the model

$$Y^* = W_0 + W_1 \ X$$

$$Y = \begin{cases} 1 \quad if \quad\quad Y^* \geq 0 \\ \\ 0 \quad\quad\quad\quad otherwise \end{cases}$$

where the two dimensional random vector (W_0, W_1) has a Normal distribution with mean (μ_0, μ_1) and variance covariance

$$\Omega = \begin{pmatrix} \omega_{00} & \omega_{01} \\ \omega_{10} & \omega_{11} \end{pmatrix}.$$

For questions (a)-(e), assume that X is a discrete random variable, distributed independently of (W_0, W_1) and with support $\{0, 1\}$. Denote the probability that $X = 1$ by p.

For questions (b)-(e), the derived expressions must be in terms of the parameters given above.

(a; 2) Is X independent of W_1, conditional on W_0? Provide a proof for your answer.

(b; 2) What is the probability density of Y^* conditional on $X = 1$? Explain.

(c; 8) What is the (unconditional) variance of Y^*? Explain

(d; 2) What is the probability that $X = 1$ conditional on $Y = 1$?

(e; 4) What is the Moment Generating Function of Y? Explain.

(f; 8) Suppose that the model is as described above, except that instead of X being discrete with support $\{0,1\}$, X is distributed $N\left(\mu_X, \sigma_X^2\right)$, independently of (W_0, W_1). Suppose also that (i) $\mu_0 = 0$, $\omega_{01} = \omega_{10} = 0$, and that (ii) (Y, X) is observable while (W_0, W_1, Y^*) is unobservable. Determine the identified parameters in the set $\{\mu_X,\ \sigma_X^2,\ \mu_1,\ \omega_{00},\ \omega_{11}\}$. Provide proofs for your answers.

(g; 4) Suppose that the model is

$$Y^* = -g\left(W_0\right) + X$$

$$Y = \begin{cases} 1 \quad if \quad & Y^* \geq 0 \\ \\ 0 & otherwise \end{cases}$$

where W_0 is distributed $N(0,1)$, X is distributed $N\left(\mu_X, \sigma_X^2\right)$, independently of W_0, and $g : R \to R$ is a continuous and strictly increasing function. Similarly to (f), suppose that (Y, X) is observable while (W_0, Y^*) is unobservable. Is the function g identified? Provide a proof for your answer.

Part II - 203B

Instruction for Part II: Solve every question.

Grading policy for Part II: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let T denote the total number of points.

1. If $T \geq 25$, you will get H.
2. If $20 \leq T < 25$, you will get P.
3. If $15 \leq T < 20$, you will get M.
4. If $T < 15$, you will get F.

Question 1 (5 pts.) *No derivation is needed for this question; your derivation will not be read anyway.* Suppose that you are given Classical Linear Regression Model I ($y = X\beta + \varepsilon$, X is a nonstochastic matrix, X has a full column rank, $E[\varepsilon] = 0$, and $E[\varepsilon\varepsilon'] = \sigma^2 I_n$ for some unknown positive number σ^2), and the data

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & -5 \\ 1 & 5 \\ 1 & -4 \end{bmatrix}, \quad y = \begin{bmatrix} 4 \\ 6 \\ -9 \\ 11 \\ -7 \end{bmatrix}.$$

Let $\left(\widehat{\beta}_1, \widehat{\beta}_2\right)'$ denote the OLS estimator. Calculate an unbiased estimator of $\operatorname{Var}\left(\widehat{\beta}_2\right)$ using the formula that you learned in class. Your answer should be numerical. A generic asymptotic formula will result in zero credit. Hint:

$$X'X = \begin{bmatrix} 5 & 0 \\ 0 & 74 \end{bmatrix}, \quad \left(X'X\right)^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{74} \end{bmatrix}, \quad X'y = \begin{bmatrix} 5 \\ 148 \end{bmatrix},$$

$$\left(X'X\right)^{-1} X'y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad X\left(X'X\right)^{-1} X'y = \begin{bmatrix} 5 \\ 5 \\ -9 \\ 11 \\ -7 \end{bmatrix}.$$

Question 2 (5 pts.) *No derivation is needed for this question; your derivation will not be read anyway.* Suppose that

$$E\left[x_i\left(y_i - x_i\theta\right)\right] = 0$$
$$E\left[z_i\left(y_i - x_i\theta\right)\right] = 0$$

Let $\widehat{\theta}$ denote the optimal (two step) GMM estimator based on the two moments above. Suppose that $(y_i, x_i, z_i)'$ $i = 1, 2, \ldots$ are i.i.d. Under standard regularity conditions, $\sqrt{n}\left(\widehat{\theta} - \theta\right)$ converges in distribution to $N(0, \Omega)$ for some Ω. Assume further that

$$\begin{bmatrix} x_i \\ z_i \\ \varepsilon_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right),$$

where $\varepsilon_i = y_i - x_i\theta$. What is the numerical value of Ω?

Question 3 (5 pts.) *No derivation is needed for this question. Just state your answer.* Consider a linear model

$$y_i = x_i\beta + \varepsilon_i$$

with the restriction that

$$E[z_i\varepsilon_i] = 0$$

where $\dim(\beta) = 1$ and $\dim(z_i) = 2$. Assume that

$$x_i = z_i'\pi + v_i$$

$$\begin{bmatrix} z_i \\ v_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right),$$

$$\pi = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and that the conditional distribution of ε_i given z_i is $N\left(0, z_{i2}^2\right)$, where z_{i2} denotes the second component of z_i. State the asymptotic distribution of $\sqrt{n}\left(\widehat{\beta} - \beta\right)$, where $\widehat{\beta}$ denotes the 2SLS

$$\begin{aligned}\widehat{\beta} &= \left[\left(\sum_i x_i z_i'\right) \left(\sum_i z_i z_i'\right)^{-1} \left(\sum_i z_i x_i'\right) \right]^{-1} \left(\sum_i x_i z_i'\right) \left(\sum_i z_i z_i'\right)^{-1} \left(\sum_i z_i \cdot y_i\right) \\ &= \left[X'Z\left(Z'Z\right)^{-1} Z'X\right]^{-1} X'Z\left(Z'Z\right)^{-1} Z'y.\end{aligned}$$

Your answer should be numerical. A generic asymptotic formula will result in zero credit. Hint: You may want to recall that the third and fourth moments of $N(0, 1)$ are 0 and 3, respectively. Also, you may use the fact that

$$\left[\left(\frac{1}{n}\sum_i x_i z_i'\right) \left(\frac{1}{n}\sum_i z_i z_i'\right)^{-1} \left(\frac{1}{n}\sum_i z_i x_i'\right) \right]^{-1} \left(\frac{1}{n}\sum_i x_i z_i'\right) \left(\frac{1}{n}\sum_i z_i z_i'\right)^{-1}$$

converges to $\begin{bmatrix} \frac{1}{6} & \frac{1}{3} \end{bmatrix}$ in probability.

Question 4 (5 pts.) *No derivation is needed for this question; your derivation will not be read anyway.* Your answers should be numerical. The only symbols allowed in your answers are n and ρ. A generic asymptotic formula will result in zero credit.

Suppose that you are given a model that satisfies following assumptions:

- $y_i = \alpha + x_{i1}\beta_1 + x_{i2}\beta_2 + \varepsilon_i$ $(i = 1, \ldots, n)$ with β unknown.
- (x_{i1}, x_{i2}) are nonstochastic. Furthermore, it is known

$$\sum_{i=1}^{n} x_{i1} = 0, \quad \sum_{i=1}^{n} x_{i2} = 0,$$
$$\sum_{i=1}^{n} x_{i1}^2 = 1, \quad \sum_{i=1}^{n} x_{i2}^2 = 1, \quad \sum_{i=1}^{n} x_{i1}x_{i2} = \rho.$$

- ε_i are independent of each other such that $\varepsilon_i \sim N\left(0, \sigma^2\right)$.

Let

$$\left(\widehat{\alpha}, \widehat{\beta}_1, \widehat{\beta}_2\right) = \underset{a,b_1,b_2}{\operatorname{argmin}} \sum_{i=1}^{n} \left(y_i - a - x_{i1}b_1 - x_{i2}b_2\right)^2$$

(a) (1 pt.) What is the variance of $\widehat{\beta}_1$?

(b) (1 pt.) What is the variance of $\widehat{\beta}_2$?

(c) (2 pt.) What is the covariance between $\widehat{\beta}_1$ and $\widehat{\beta}_2$?

(d) (1 pt.) For what value of ρ are $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are independent of each other?

Question 5 (5 pts.) *No derivation is needed for this question; your derivation will not be read anyway.* Suppose that

$$y_i = \alpha + \beta x_i + \varepsilon_i$$

with $(y_i, x_i)'$ $i = 1, \ldots, n$ i.i.d., and $E[\varepsilon_i] = E[z_i\varepsilon_i] = 0$. You are given the following data set with $n = 3$:

$$\begin{bmatrix} y_1 & x_1 & z_1 \\ y_2 & x_2 & z_2 \\ y_3 & x_3 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & 0 \\ 1 & 1 & -2 \end{bmatrix}$$

Provide 95% asymptotic confidence interval for β, based on the IV estimate of $(\alpha, \beta)'$. Do NOT assume that $E\left[\varepsilon_i^2 \middle| z_i\right]$ is a constant. Note that β is the second component of the vector $(\alpha, \beta)'$, i.e., it is a <u>scalar</u>. If your answer involves a square root, try to simplify as much as you can. If you simply provide a generic and abstract asymptotic variance/confidence interval formula, your answer will be judged to be incorrect.

Hint: We have

$$\left[\sum_{i=1}^{n} \begin{pmatrix} 1 \\ z_i \end{pmatrix} \begin{pmatrix} 1 & x_i \end{pmatrix}\right]^{-1} \left[\sum_{i=1}^{n} \begin{pmatrix} 1 \\ z_i \end{pmatrix} y_i\right]^{-1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} - \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix},$$

and

$$\sum_{i=1}^{n}\begin{pmatrix}1\\ z_i\end{pmatrix}\begin{pmatrix}1 & x_i\end{pmatrix}$$
$$=\begin{bmatrix}1\\ 2\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}+\begin{bmatrix}1\\ -2\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}$$
$$=\begin{bmatrix}3 & 0\\ 0 & -4\end{bmatrix}.$$

Question 6 (5 pts.) *No derivation is needed for this question; your derivation will not be read anyway.* Suppose we want to estimate a model

$$y_i = \beta_1 + \beta_2 \cdot x_{i2} + \beta_3 \cdot x_{i3} + \varepsilon_i, \quad i = 1, \ldots, n.$$

Suppose that the model satisfies the assumptions of classical linear regression model II. You estimated $(\beta_1, \beta_2, \beta_3)'$ by OLS. Your computer reported

$$\begin{bmatrix}\widehat{\beta}_1\\ \widehat{\beta}_2\\ \widehat{\beta}_3\end{bmatrix}=\begin{bmatrix}1.2\\ 0.15\\ 0.35\end{bmatrix}.$$

The number of observations (n) is equal to 33, and the sum of squared residuals ($e'e$) is equal to 0.2. Your computer also reported the estimated variance covariance matrix $\widehat{\mathbb{V}} = s^2 (X'X)^{-1}$ of $\left(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3\right)'$:

$$\begin{bmatrix}(0.4)^2 & \frac{1}{2}(0.4)(0.1) & \frac{1}{3}(0.4)(0.5)\\ \frac{1}{2}(0.1)(0.4) & (0.1)^2 & \frac{1}{2}(0.1)(0.5)\\ \frac{1}{3}(0.5)(0.4) & \frac{1}{2}(0.5)(0.1) & (0.5)^2\end{bmatrix},$$

where s^2 is the unbiased estimate of $E\left[\varepsilon_i^2\right]$ based on $e'e$, as discussed in class.

- Test the hypothesis $H_0 : \beta_2 = 0.05$ against $H_1 : \beta_2 \neq 0.05$ under 5% significance level.
  (a) (1 pt.) State the numerical value of your t-statistic.
  (b) (1 pt.) State the degrees of freedom of the t-distribution (the distribution of the t-statistic under the null). Your answer should be a concrete number.
  (c) (1 pt.) Do you reject the null or do you accept the null? (Assume that the t-distribution is so close to the standard normal distribution that the difference can be ignored when characterizing the critical value.)
- (2 pts.) Provide the 95% confidence interval for

$$5\beta_2 + \beta_3$$

  using the formula you learned in class. (Again, ignore the difference of the t-distribution and the standard normal distribution.) Your answer should be numerical; an abstract formula will not be accepted as an answer.

Part III - 203C

Instruction for Part III: Solve every question.

Grading policy for Part III: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let T denote the total number of points.

1. If $T \geq 25$, you will get H.
2. If $20 \leq T < 25$, you will get P.
3. If $15 \leq T < 20$, you will get M.
4. If $T < 15$, you will get F.

Question 1 (8 points) Let X have the probability mass function $f(x;\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$, zero elsewhere. We test $H_0 : \theta = \frac{1}{2}$ against $H_1 : \theta < \frac{1}{2}$ by taking an random (i.i.d.) sample $X_1, X_2, \ldots, X_5$ of size $n = 5$ and rejecting H_0 if $Y = \sum_{i=1}^n X_i$ is observed to be less than or equal to a constant c.

(a) (4 points) Is this a uniformly most powerful test? Justify your answer.

(b) (2 points) Find the size of the test when $c = 1$.

(c) (2 points) Find the size of the test when $c = 0$.

Question 2 (10 points) Let $\{X_t\}_{t\in\mathbb{Z}}$ be a time series generated by

$$X_t = u_t + \theta_1 u_{t-1} + \theta_2 u_{t-2}, \text{ where } \{u_t\}_{t\in\mathbb{Z}} \sim WN(0, \sigma^2),$$

where θ_1 and θ_2 are finite real numbers. Let $\gamma_X(h)$ and $\rho_X(h)$ ($h \in \mathbb{Z}$) denote the auto-covariance function and the auto-correlation function of $\{X_t\}_{t\in\mathbb{Z}}$ respectively.

(a) (4 points) What are the largest and smallest possible values for $\rho_X(1)$?

(b) (3 points) What are the largest and smallest possible values for $\rho_X(2)$?

(c) (3 points) Suppose that $\sigma^2 = 1$. Find the values of θ_1 and θ_2 such that $\gamma_X(0) = 2$, $\gamma_X(1) = 0$, $\gamma_X(2) = -1$, and $\gamma_X(h) = 0$ for $|h| > 2$.

Question 3 (12 points) Consider the ARMA(1,1) model

$$X_t = \phi X_{t-1} + u_t + \theta u_{t-1}$$

where $|\phi| < 1$, θ is a finite constant and $\{u_t\}_t$ is an $IID(0, \sigma^2)$ process. Let $\gamma_X(h)$ and $\rho_X(h)$ ($h \in \mathbb{Z}$) denote the auto-covariance function and the auto-correlation function of $\{X_t\}_{t\in\mathbb{Z}}$ respectively.

(a) (3 points) Find the auto-covariance function of $\{X_t\}_t$.

(b) (3 points) Find the long-run variance of $\{X_t\}_t$.

(c) (4 points) Suppose we know that $\gamma_X(0) = 4$, $\rho_X(1) = 3/4$ and $\rho_X(2) = 3/8$. Find the values of ϕ, θ and σ^2.

(d) (2 points) Suppose we know that $\gamma_X(0) = 4$, $\rho_X(1) = 3/4$ and $\rho_X(2) = 3/8$. Find the value of the long-run variance of $\{X_t\}_t$.