Department of Economics
UCLA

Spring 2015

Comprehensive Examination
Quantitative Methods

This exam consists of three parts. You are required to answer all the questions in all the parts. Each part is worth 100 points, with relative weights given by the points for each question. Allocate your time wisely. Good luck!

Part I - 203A

Question 1

Consider the following situation describing the relationship among random variables Z, λ, γ, Y_1 and Y_2. The density f_Z of Z is given by

$$f_Z(z) = \frac{e^{-\left(\frac{1}{2}\right)\left(\frac{z-\mu}{\sigma}\right)^2}}{\sqrt{2\pi\sigma^2}} \qquad -\infty < z < \infty.$$

The probability mass functions of λ and of γ are given by

$$\lambda = \begin{cases} 1 & if \ \ Z \leq 0 \\ 2 & otherwise \end{cases}$$

and

$$\gamma = \begin{cases} 1 & if \ \ Z \leq 2 \\ 2 & otherwise \end{cases}$$

The conditional joint density of (Y_1, Y_2), given (λ, γ), is

$$f_{(Y_1,Y_2)|(\lambda,\gamma)}(y_1, y_2) = \begin{cases} \lambda\,\gamma\, e^{-\lambda\, y_1 - \gamma\, y_2} & if \ \ y_1 > 0,\ y_2 > 0 \\ 0 & otherwise \end{cases}$$

(a; 5 points) Derive the joint probability of (λ, γ). Are λ and γ independently distributed? Provide a proof.

(b; 10 points) Derive an expression for the joint density of (Y_1, Y_2) in terms of μ and σ.

(c; 10 points) Provide an expression for the probability that $(Y_1 \geq 1)$ in terms of only μ and σ.

(d; 5 points) If it is known that $(Y_1 \geq 1)$, what is the probability that $(\lambda = 1)$? Explain.

(c; 10 points) (i)Are Y_1 and Y_2 independently distributed? (ii) Are Y_1 and Y_2 conditionally independent, given (λ, γ)? (iii) Are Y_1 and Y_2 conditionally independent, given Z? Justify your answers.

(f; 5 points) Derive an expression, in terms of only μ and σ, for the conditional expectation of Y_2 given $Y_1 = 1$.

(g; 20 points) Let $W_1 = \min\{Y_1, Y_2\}$, $W_2 = \max\{Y_1, Y_2\}$, and $T = (W_1 + W_2)^{1/2}$. Derive the conditional density of T, given $(\lambda, \gamma) = (2, 2)$.

Question 2

Suppose that X is a continuous random variable with an everywhere positive density and that the distribution of the random variable Y conditional on X is given, for all x, by

$$f_{Y|X=x}(y) = \frac{1}{\sqrt{2\pi\ \sigma(x)^2}} \exp^{-\frac{1}{2}\left(\frac{y-m(x)}{\sigma(x)}\right)^2} \qquad -\infty < y < \infty$$

where $m(x)$ and $\sigma(x)$ are unknown functions of x. Let Z be defined by

$$Z = \begin{cases} 0 & if \quad Y \leq a \\ 1 & if \quad a < Y \leq b \\ 2 & otherwise \end{cases}.$$

where a and b are constants of unknown values such that $a < b$.

(a; 5 points) What is the conditional probability of Z given X?

(b; 5 points) Are the functions m and σ identified? Justify your answer.

(c; 5 points) If $\sigma(x) = 1$ for all x, is $(b - a)$ identified? Justify your answer.

Let T be an observable random variable, possessing an everywhere positive density. Let W be defined by

$$W = T + Y.$$

(d; 20 points) Suppose that you can only observe X, T, Z, and the conditional expectation of W given (X, T). For each of the unknown functions and parameters, $m(x)$, $\sigma(x)$, a and b, determine whether it is identified. Justify your answers.

Part II - 203B

Question 1 (40 pts.)

Suppose that

$$y_{1i} = x_{1i}\beta_1 + u_{1i}$$
$$y_{2i} = x_{2i}\beta_2 + u_{2i}$$

We assume that (i) (x_{1i}, x_{2i}) is independent of (u_{1i}, u_{2i}); (ii) u_{1i} and u_{2i} are independent of each other; (iii) $E[u_{1i}] = E[u_{2i}] = 0$, $\mathrm{Var}(u_{1i}) = 5$, $\mathrm{Var}(u_{2i}) = 3$, and $E[x_{1i}^2] = E[x_{2i}^2] = 1$; and (iv) we observe $(y_{1i}, y_{2i}, x_{1i}, x_{2i})$ $i = 1, \ldots, n$, which are assumed to be i.i.d.

Propose an estimator $\widehat{\theta}$ of $\theta = \beta_1 - \beta_2$. Derive and characterize the asymptotic distribution of $\sqrt{n}\left(\widehat{\theta} - \theta\right)$.

Note 1: Your characterization of the asymptotic distribution should be such that the asymptotic variance is a concrete number; an abstract formula is not acceptable as an answer.
Note 2: Many of you will work with some obvious estimators $\widehat{\beta}_1$ and $\widehat{\beta}_2$ of β_1 and β_2. You would have to establish the joint distribution of $\sqrt{n}\left(\widehat{\beta}_1 - \beta_1\right)$ and $\sqrt{n}\left(\widehat{\beta}_2 - \beta_2\right)$.
Note 3: You are allowed to use the law of large numbers, central limit theorem, delta method, and Slutsky theorem. You may also use the following result:

Theorem 1 *Suppose that*

$$E\left[\underset{q\times 1}{Y_i} - \underset{q\times q}{X_i}\ \underset{q\times 1}{\theta}\right] = 0$$

Suppose that (Y_i, X_i) $i = 1, 2, \ldots$ *is i.i.d. Also suppose that* $G_0 = E[X_i]$ *is nonsingular. Finally, suppose that* $S_0 = E[U_i U_i']$ *exists and is finite. Then,*

$$\widehat{\theta} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \left(\sum_{i=1}^{n} X_i\right)^{-1}\left(\sum_{i=1}^{n} Y_i\right)$$

is such that

$$\sqrt{n}\left(\widehat{\theta} - \theta\right) \to N\left(0, G_0^{-1} S_0 \left(G_0'\right)^{-1}\right)$$

Question (40 pts.)

Consider a model

$$y_i = \alpha + \beta_i x_i + \varepsilon_i$$

We assume that $(x_i, \beta_i, \varepsilon_i)'$ is iid. We assume that (i) x_i is independent of ε_i; and (ii) ε_i has a mean zero. We observe $(y_i, x_i)'$ for each individual i. Let $\left(\widehat{\alpha}, \widehat{\beta}\right)'$ denote the OLS estimator when y_i are regressed on $(1, x_i)'$ for $i = 1, \ldots, n$.

Derive the probability limit of $\left(\widehat{\alpha}, \widehat{\beta}\right)'$, and propose a sufficient condition under which $\operatorname{plim} \widehat{\beta} = E\left[\beta_i\right]$. (You are NOT allowed to assume that β_i is nonstchastic.)

Question (20 pts.)

Consider the following two-equation model:

$$
\begin{aligned}
y_1 &= \gamma y_2 + \beta_{11} x_1 + \beta_{21} x_2 + \beta_{31} x_3 + \varepsilon_1 \\
y_2 &= \gamma y_1 + \beta_{12} x_1 + \beta_{22} x_2 + \varepsilon_2
\end{aligned}
$$

where we assume that (x_1, x_2, x_3) is independent of $(\varepsilon_1, \varepsilon_2)$ and $E\left[\varepsilon_1\right] = E\left[\varepsilon_2\right] = 0$.

Show how β_{11} can be identified.

Note 1: You would have to assume that certain matrices are nonsingular. Explicitly state which matrices need to be nonsingular.
Note 2: This question is NOT about the order condition, i.e., if you try to answer this question by simply counting the numbers of certain kinds of variables, you will get zero credit.

Part III - 203C

1. (a) (5 points) We have one observation X_1 from the normal distribution:

$$X_1 \sim N(\theta, 1), \text{ where } \theta \in (-\infty, +\infty).$$

Find the level-α UMP test $\varphi_{1,\alpha}$ for testing $H_0 : \theta = 0$ v.s. $H_1 : \theta > 0$.

(b) (5 points) We have one observation (X_1, X_2) from the normal distribution:

$$(X_1, X_2)' \sim N\left(\begin{pmatrix} \theta \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\right).$$

Find the level-α UMP test $\varphi_{2,\alpha}$ for testing $H_0 : \theta = 0$ v.s. $H_1 : \theta > 0$.

(c) (5 points) Compare the power functions of the tests $\varphi_{1,\alpha}$ and $\varphi_{2,\alpha}$, and explain your finding.

(d) (5 points) We have one observation (X_1, X_2) from the normal distribution:

$$(X_1, X_2)' \sim N\left(\begin{pmatrix} \theta \\ \gamma \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\right)$$

where θ and γ are unknown. Construct a level α test for testing $H_0 : \gamma = 0$ v.s. $H_1 : \gamma \neq 0$.

(e) (10 points) We are interested in testing

$$H_0 : \theta = 0 \text{ v.s. } H_1 : \theta > 0.$$

Under the same conditions of (d), is it possible to combine the tests in (a), (b) and (d) to get a more powerful level-α test than $\varphi_{1,\alpha}$ in (a)? Justify your answer.

2. Consider a time series regression with $AR(1)$ error

$$\begin{aligned} Y_t &= X_t \beta + u_t \\ u_t &= \rho u_{t-1} + \varepsilon_t \end{aligned}$$

where ε_t are $i.i.d.(0, \sigma_\varepsilon^2)$ with finite 4-th moment, $\{X_t\}$ are i.i.d.$(0, \sigma_X^2)$ with finite 4-th moment, X_t is independent with respect to ε_s for any t and s.

(a) (10 points) Suppose that $|\rho| < 1$. Consider the LS estimator

$$\widehat{\beta}_T = \frac{\sum_{t=1}^T X_t Y_t}{\sum_{t=1}^T X_t^2}.$$

Derive the asymptotic distribution of $\widehat{\beta}_T$.

(b) (10 points) Let $\widehat{u}_t = Y_t - X_t\widehat{\beta}_T$ and $\widehat{\rho}_T = \sum_{t=2}^T \widehat{u}_t\widehat{u}_{t-1} \Big/ \sum_{t=1}^T \widehat{u}_t^2$ be the LS estimator of ρ based on the regression of $\widehat{u}_t$ on $\widehat{u}_{t-1}$. Under the same conditions in (a), show that $\widehat{\rho}_T$ is a root-T consistent estimator of ρ.

(c) (15 points) We can construct an estimator $\widehat{\beta}_T^*$ by regressing $\widehat{Y}_t$ on X_t, where $\widehat{Y}_t = Y_t - \widehat{\rho}_T(Y_{t-1} - X_{t-1}\widehat{\beta}_T)$. That is

$$\widehat{\beta}_T^* = \frac{\sum_{t=2}^T X_t\widehat{Y}_t}{\sum_{t=1}^T X_t^2}.$$

Under the same conditions in (a), derive the asymptotic distribution of $\widehat{\beta}_T^*$.

(d) (5 points) Compare the asymptotic variances of $\widehat{\beta}_T$ and $\widehat{\beta}_T^*$ you get in (a) and (c), and discuss your findings.

(e) (10 points) Suppose that $\rho \in [0, 1]$. Consider the LS estimator of β based on the detrend data

$$\widetilde{\beta}_T = \frac{\sum_{t=2}^T \Delta X_t \Delta Y_t}{\sum_{t=2}^T \Delta X_t^2},$$

where $\Delta X_t = X_t - X_{t-1}$ and $\Delta Y_t = Y_t - Y_{t-1}$. Derive the asymptotic distribution of $\widetilde{\beta}_T$.

(f) (5 points) Compare the asymptotic properties of $\widehat{\beta}_T$ and $\widetilde{\beta}_T$, and discuss your findings.

(g) (15 points) Suppose that $\rho \in [0, 1]$. Construct a test on whether u_t is a unit root process. Show that your test has the asymptotic size α and asymptotic power 1 against any fixed alternative.

Some Useful Theorems and Lemmas

The joint density of bivariate normal random variable

$$(X_1, X_2)' \sim N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \sigma_{1,2}\\ \sigma_{1,2} & \sigma_2^2\end{pmatrix}\right)$$

is

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{z}{2(1-\rho^2)}\right)$$

where $\rho = \frac{\sigma_{1,2}}{\sigma_1\sigma_2}$ and

$$z = \frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}.$$

Theorem 2 (Martingale Convergence Theorem) *Let $\{(X_t, \mathcal{F}_t)\}_{t\in\mathbb{Z}_+}$ be a martingale in L^2. If $\sup_t E\left[|X_t|^2\right] < \infty$, then $X_n \to X_\infty$ almost surely, where X_∞ is some element in L^2.*

Theorem 3 (Martingale CLT) *Let $\{X_{t,n}, \mathcal{F}_{t,n}\}$ be a martingale difference array such that $E[|X_{t,n}|^{2+\delta}] < \Delta < \infty$ for some $\delta > 0$ and for all t and n. If $\overline{\sigma}_n^2 > \delta_1 > 0$ for all n sufficiently large and $\frac{1}{n}\sum_{t=1}^n X_{t,n}^2 - \overline{\sigma}_n^2 \to_p 0$, then $n^{\frac{1}{2}}\overline{X}_n/\overline{\sigma}_n \to_d N(0,1)$.*

Theorem 4 (LLN of Linear Processes) *Suppose that Z_t is i.i.d. with mean zero and $E[|Z_0|] < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k|\varphi_k| < \infty$. Then $n^{-1}\sum_{t=1}^n X_t \to_{a.s.} 0$.*

Theorem 5 (CLT of Linear Processes) *Suppose that Z_t is i.i.d. with mean zero and $E[Z_0^2] = \sigma_Z^2 < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k^2\varphi_k^2 < \infty$. Then $n^{-\frac{1}{2}}\sum_{t=1}^n X_t \to_d N\left[0, \varphi(1)^2\sigma_Z^2\right]$.*

Theorem 6 (LLN of Sample Variance) *Suppose that Z_t is i.i.d. with mean zero and $E[Z_0^2] = \sigma_Z^2 < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k\varphi_k^2 < \infty$. Then*

$$\frac{1}{n}\sum_{t=1}^n X_t X_{t-h} \to_p \Gamma_X(h) = E\left[X_t X_{t-h}\right]. \tag{1}$$

Theorem 7 (Donsker) *Let $\{u_t\}$ be a sequence of random variables generated by $u_t = \sum_{k=0}^\infty \varphi_k \varepsilon_{t-k} = \varphi(L)\varepsilon_t$, where $\{\varepsilon_t\} \sim iid\,(0, \sigma_\varepsilon^2)$ with finite fourth moment and $\{\varphi_k\}$ is a sequence of constants with $\sum_{k=0}^\infty k|\varphi_k| < \infty$. Then $B_{u,n}(\cdot) = n^{-\frac{1}{2}}\sum_{t=1}^{[n\cdot]} u_t \to_d \lambda B(\cdot)$, where $\lambda = \sigma_\varepsilon\varphi(1)$.*