Comprehensive Examination
Quantitative Methods
Fall, 2016

The exam consists of three parts. You are required to answer all the questions in all the parts.

Part I - 203A

Some potentially useful facts/hints can be found at the end.

1. (5 pts.) *No derivation is required for this question; your derivation will not be read anyway.* The joint pdf of random variables X and Y is given by

$$f_{Y,X}(y,x) = \begin{cases} c & \text{if } 0 < y < 1,\ y > x > 0 \\ 0 & \text{otherwise} \end{cases}$$

for some c. What is $E[Y|\, x]$ for $x = \frac{1}{3}$? Your answer should be a number.

2. (5 pts.) *No derivation is required for this question; your derivation will not be read anyway.* Suppose that $Y_n \sim \chi^2(n)$. What is the asymptotic distribution of $\sqrt{Y_n} - \sqrt{n}$ as $n \to \infty$? Your answer should be numerical; an abstract formula will not be accepted as an answer.

3. (2 pts.) *No derivation is required for this question; your derivation will not be read anyway.* Let $F_{Y|X}(y|\, x)$ denote the conditional CDF of Y given X, i.e., $F_{Y|X}(y|\, x) = P(Y \le y|\, x)$. Suppose that $F_{Y|X}(y|\, x)$ is continuous and strictly increasing in y for all x in the support of X. Also suppose that $X \sim N(0,1)$. What is $E\left[F_{Y|X}(Y|\, X)\, X^2\right]$? Your answer should be a number.

4. (3 pts.) *No derivation is required for this question; your derivation will not be read anyway.* Let G denote the CDF of $N(1,4)$, and let G^{-1} denote its inverse. Suppose that V have a uniform distribution over $(0,1)$. Calculate $E\left[\left(G^{-1}(V)\right)^2\right]$.

5. (5 pts.) *No derivation is required for this question; your derivation will not be read anyway.* Suppose that $X_1, \ldots, X_n$ are i.i.d. Their common distribution is the uniform distribution over the interval $(0, \theta_0)$, i.e., the common PDF is equal to $1/\,\theta_0$ over $(0, \theta_0)$, and 0 elsewhere. Let $\widehat{\theta}$ denote the MLE of θ_0. Let $x > 0$ be given. Derive the limit of $\Pr\left[n\left(\theta_0 - \widehat{\theta}\right) \le x\right]$ assuming that $\theta_0 = 1$.

6. (5 pts.) *No derivation is required for this question; your derivation will not be read anyway.* Suppose that

$$f(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

It can be shown that (i) the $f(\cdot)$ is a valid PDF; and (ii) $f(x) \le 2$. Let (U_i, V_i) $i = 1, 2, \ldots$ denote an i.i.d. sequence of random vectors such that (i) U_i and V_i are uniform$(0,1)$ random variables; and (ii) U_i and V_i are independent of each other. What is the probability limit of

$$\frac{\frac{1}{n}\sum_{i=1}^{n} V_i \cdot 1\left(U_i \le \frac{1}{2} f(V_i)\right)}{\frac{1}{n}\sum_{i=1}^{n} 1\left(U_i \le \frac{1}{2} f(V_i)\right)}$$

as $n \to \infty$? Your answer should be a number.

7. (5 pts.) *In this question, your derivation/argument will be read and evaluated.* Suppose that

$$Y = \alpha + U_1 X + U_2,$$

where X is independent of (U_1, U_2). We observe (Y, X) but (U_1, U_2) is unobserved. Suppose that it is known that

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \right).$$

Also suppose that the support of X is $\{1, 2\}$, i.e., for some p with $0 < p < 1$, we have

$$P(X = 1) = p$$
$$P(X = 2) = 1 - p$$

Which of the parameters $\alpha, \mu_1, \sigma_1^2, \sigma_2^2$ are identified? Why?

Potentially useful facts/hints

1. We define $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$. It is known that (i) $\Gamma(1) = \int_0^\infty e^{-y} dy = 1$; and (ii) if α is a positive integer greater than 1, $\Gamma(\alpha) = (\alpha - 1)!$

2. If X has a PDF

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

then

$$E[X] = \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

3. **Hint for Question 5**: The likelihood is

$$L(\theta) = \prod_{i=1}^n \frac{1(0 < X_i < \theta)}{\theta} = \frac{1(\max_{1\le i\le n} X_i \le \theta)}{\theta^n}$$

Note that for $\theta \in (0, \max_{1\le i\le n} X_i)$, we have $L(\theta) = 0$. Also note that for $\theta \in [\max_{1\le i\le n} X_i, \infty)$, we have $L(\theta) = \theta^{-n}$ monotonically decreasing in θ.

Part II - 203B

Some potentially useful facts/hints can be found at the end.

1. (5 pts.) *No derivation is required for this question; your derivation will not be read anyway.* Suppose that
$$y_i = \alpha + \beta x_i + \varepsilon_i$$
with $(y_i, x_i)'$ $i = 1, \ldots, n$ i.i.d., and $E[\varepsilon_i] = E[z_i \varepsilon_i] = 0$. You are given the following data set with $n = 3$:
$$\begin{bmatrix} y_1 & x_1 & z_1 \\ y_2 & x_2 & z_2 \\ y_3 & x_3 & z_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 4 & 1 & -1 \end{bmatrix}$$
The IV estimate of $(\alpha, \beta)'$ using $(1, z_i)'$ as IV is
$$\left[\sum_{i=1}^{n} \begin{pmatrix} 1 \\ z_i \end{pmatrix} \begin{pmatrix} 1 & x_i \end{pmatrix}\right]^{-1} \left[\sum_{i=1}^{n} \begin{pmatrix} 1 \\ z_i \end{pmatrix} y_i\right]^{-1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
Provide 95% asymptotic confidence interval for β. Do NOT assume that $E[\varepsilon_i^2 | z_i]$ is a constant. If your answer involves a square root, try to simplify as much as you can. If you simply provide a generic and abstract asymptotic variance/confidence interval formula, your answer will be judged to be incorrect.

2. (2 pts.) *No derivation is required for this question; your derivation will not be read anyway.* Suppose that $y = X\beta + \varepsilon$ such that X is an $n \times 2$ nonstochastic matrix with full column rank, and $\varepsilon \sim N(0, \sigma^2 I_n)$. The first column of X consists of 1's, i.e., we can write $y_i = \beta_1 + \beta_2 x_{i2} + \varepsilon_i$ for $i = 1, \ldots, n$. Suppose that $\mathbb{R}^2 = 0.1$, and $n = 83$. Test $H_0 : \beta_2 = 0$ against $H_1 : \beta_2 \neq 0$ at the 5% significance level (α), by deriving the t-statistic from the given set of information. (Assume that the t-distribution is so close to the standard normal distribution that the difference can be ignored when characterizing the critical value.) Your answer should include (i) the numerical value of the absolute value of the t-statistic; and (ii) the result of the test, i.e., whether you reject the null or accept it.

3. *No derivation is required for the questions below; your derivation will not be read anyway.* In the questions below, suppose that X is a 10×3 matrix with full column rank, and that $y = X\beta + \varepsilon$ where $\varepsilon \sim N(0, I_{10})$. Let $\widehat{\beta} = (X'X)^{-1} X'y$.

   (a) (2 pts.) What is $E\left[\left(y - X\widehat{\beta}\right)'\left(y - X\widehat{\beta}\right)\right]$? Your answer should be a number.

   (b) (2 pts.) What is $\text{Var}\left(\left(y - X\widehat{\beta}\right)'\left(y - X\widehat{\beta}\right)\right)$? Your answer should be a number.

4. *In the questions below, your argument will be read and evaluated. You are expected to use only the order condition, i.e., you are expected to compare the number of included endogenous variables on the RHS and the number of excluded exogenous variables and*

draw the conclusion. Do not waste time deriving the order condition; it is sufficient that you know how to use it. Also, you do not have to use the rank condition (i.e., discussion of nonsingularity of certain matrices). You are given a sample produced produced by a simultaneous equations model:

$$
\begin{aligned}
y_1 &= \alpha_1 y_2 + \alpha_2 x + \varepsilon_1 \\
y_2 &= \alpha_3 y_1 + \varepsilon_2
\end{aligned}
$$

where the only moment restriction is that $E[x\varepsilon_1] = E[x\varepsilon_2] = 0$.

(a) (2 pts.) Is $(\alpha_1, \alpha_2)'$ identified?

(b) (2 pts.) Is α_3 identified?

5. (5 pts.) *No derivation is required for this question; your derivation will not be read anyway.* Suppose that

$$
\begin{aligned}
y_i &= x_i \beta + \varepsilon_i \\
x_i &= z_{1i} + 0 \times z_{2i} + v_i
\end{aligned}
$$

where (ε_i, v_i) are independent of $z_i = (z_{1i}, z_{2i})'$, and $E[\varepsilon_i] = E[v_i] = 0$, $E[\varepsilon_i^2] = 2$, $E[v_i^2] = 4$, $E[\varepsilon_i v_i] = 1$, $E[z_{1i}^2] = E[z_{2i}^2] = 2$, and $E[z_{1i} z_{2i}] = 1$. Let

$$
\widehat{\beta}_1 = \frac{\sum_{i=1}^n z_{1i} y_i}{\sum_{i=1}^n z_{1i} x_i}, \quad \widehat{\beta}_2 = \frac{\sum_{i=1}^n z_{2i} y_i}{\sum_{i=1}^n z_{2i} x_i}.
$$

Under the standard assumption, i.e., $(z_{1i}, z_{2i}, \varepsilon_i, v_i)'$ i.i.d., what is the asymptotic distribution of

$$
\sqrt{n} \begin{pmatrix} \widehat{\beta}_{(1)} - \beta \\ \widehat{\beta}_{(2)} - \beta \end{pmatrix} ?
$$

Your asymptotic variance matrix should consist of concrete numbers. No abstract formula will be accepted as an answer.

6. (5 pts.) *In this question, your argument will be read and evaluated.* Suppose that

Assumption 1: (y_i, x_i, z_i) $i = 1, 2, \ldots$ are i.i.d.

Assumption 2: The matrix

$$
\begin{bmatrix} 1 & E[x_i] \\ E[z_i] & E[z_i x_i] \end{bmatrix}
$$

is nonsingular. Besides, $E[|y_i|] < \infty$, $E[|x_i|] < \infty$, $E[|z_i|] < \infty$, $E[|z_i y_i|] < \infty$, and $E[|z_i x_i|] < \infty$.

Define

$$
\widehat{\beta} \equiv \begin{bmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \end{bmatrix} = \left(\sum_{i=1}^n \begin{bmatrix} 1 \\ z_i \end{bmatrix} \begin{bmatrix} 1 & x_i \end{bmatrix} \right)^{-1} \sum_{i=1}^n \begin{bmatrix} 1 \\ z_i \end{bmatrix} y_i.
$$

Define $\beta \equiv (\beta_1, \beta_2)'$ to be the probability limit of $\widehat{\beta}$ as $n \to \infty$. Define $\varepsilon_i \equiv y_i - (\beta_1 + x_i \beta_2)$. Prove that $E[\varepsilon_i] = E[z_i \varepsilon_i] = 0$. You are not allowed to use any other assumption other than Assumption 1 and Assumption 2 above.

7. (5 pts.) *In this question, your derivation/argument will be read and evaluated.* Suppose that

$$y_i^* = \alpha + \beta \cdot x_i + \varepsilon_i,$$

where x_i is a deterministic scalar and $\varepsilon_i \sim N(0, \sigma^2)$. Our data consist of (y_i, x_i, D_i, z_i) $i = 1, \ldots, n$, where

$$\begin{aligned} y_i &\equiv y_i^* \cdot D_i \\ D_i &\equiv 1(y_i^* > 0) \\ z_i &\equiv \Pr[y_i = 1 | x_i]. \end{aligned}$$

Provide a consistent estimator of (α, β) NOT using the Heckman's first step. You may assume that your computer can evaluate $\phi(\cdot)$, $\Phi(\cdot)$, and $\Phi^{-1}(\cdot)$, where $\phi(\cdot)$ and $\Phi(\cdot)$ are the PDF and the CDF of $N(0,1)$, and $\Phi^{-1}(\cdot)$ is the inverse of the $\Phi(\cdot)$. (In your answer, you are NOT allowed to use the symbol λ. You may use $\phi(\cdot)$, $\Phi(\cdot)$, and $\Phi^{-1}(\cdot)$.)

Potentially useful facts/hints:

1. **Hint for Question 1**: We have

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} - \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

and

$$\begin{aligned} &\sum_{i=1}^{n} \begin{pmatrix} 1 \\ z_i \end{pmatrix} \begin{pmatrix} 1 & x_i \end{pmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \end{aligned}$$

2. Classical linear regression **model II** is such that (i) $y = X\beta + \varepsilon$, where

$$\underset{n \times 1}{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \underset{n \times k}{X} = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$$

(ii) X is a nonstochastic matrix; (iii) X has a full column rank; (iv) $\varepsilon \sim N\left(\underset{n\times 1}{0}, \sigma^2 I_n\right)$. In this model,

   (a) The t-statistic for $H_0 : r'\beta = q$ vs. $H_A : r'\beta \neq q$ is $\frac{r'\widehat{\beta} - r'\beta}{\sqrt{r'\widehat{\mathbb{V}}r}}$, where $\widehat{\mathbb{V}} = s^2 (X'X)^{-1}$

(b) The F-statistic for

$$H_0 : \underset{m\times k}{R}\beta = q \quad vs. \quad H_A : R\beta \neq q$$

is $\frac{1}{m}\left(R\widehat{\beta} - R\beta\right)'\left(R\widehat{\mathbb{V}}R'\right)^{-1}\left(R\widehat{\beta} - R\beta\right)$

(c) $\frac{\mathbb{R}^2 \big/ (k-1)}{(1-\mathbb{R}^2)/(n-k)}$

3. If $Y \sim N(0,1)$, then $E[Y] = E[Y^3] = 0$, $E[Y^2] = 1$, and $E[Y^4] = 3$.

Part III - 203C

Some potentially useful facts/hints can be found at the end.

1. (6 pts.) Suppose that we have one observation from a probability density function $f_\theta(x)$. Find a uniformly most powerful test of size $\alpha = 0.05$ (with explicit critical region) for
$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta = \theta_1$$
when

   (a) (3 pts.) $f_\theta(x) = \begin{cases} 2[\theta x + (1-\theta)(1-x)], & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$, $\theta_0 = \frac{1}{4}$ and $\theta_1 = \frac{1}{2}$.

   (b) (3 pts.) $f_{\theta_0}(x) = \begin{cases} \frac{1}{\exp(x)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ and $f_{\theta_1}(x) = \begin{cases} \frac{x^2}{2\exp(x)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$.

2. (12 pts.) Suppose that we have data $\{X_t\}_{t=1}^T$ from the following model
$$X_t = \theta u_t + (\theta - 1)u_{t-1}$$
where $\{u_t\}$ is $i.i.d.(0, \sigma_u^2)$ with finite 8-th moment and θ is a finite constant.

   (a) (3 pts.) Consider the LS estimator
$$\widehat{\rho}_T = \frac{\sum_{t=2}^T X_t X_{t-1}}{\sum_{t=2}^T X_{t-1}^2}. \tag{1}$$
   Show that the probability that $\widehat{\rho}_T \in (-0.6, 0.6)$ goes to 1 as $T \to \infty$.

   (b) (3 pts.) Derive the asymptotic distribution of $\widehat{\rho}_T$ defined in (1).

   (c) (1 pt.) Find the auto-covariance function $\Gamma_X(\cdot)$ of $\{X_t\}$.

   (d) (2 pts.) Construct a consistent estimator of the long-run variance of $\{X_t\}$. Show the consistency of your estimator.

   (e) (1 pt.) Suppose we know that $\theta > 1$. Show that θ and σ_u^2 are uniquely identified by the following equations
$$\frac{\theta^2 - \theta}{2\theta^2 - 2\theta + 1} = \frac{\Gamma_X(1)}{\Gamma_X(0)} \text{ and } (\theta^2 - \theta)\sigma_u^2 = \Gamma_X(1). \tag{2}$$

   (f) (2 pts.) Construct consistent estimators of θ and σ_u^2 using the restrictions in (2). Show the consistency of your estimators.

3. (12 pts.) Suppose that we have data $\{Y_t\}_{t=1}^T$ from the following time series model

$$Y_t = \mu t + \rho Y_{t-1} + u_t \text{ for } t > 0,$$

where $|\rho| \leq 1$, $\{u_t\}$ is $i.i.d.(0, \sigma_u^2)$ with finite 8-th moment and $Y_0 = 0$.

(a) (4 pts.) Consider the LS estimator

$$\widetilde{\rho}_T = \frac{\sum_{t=2}^T Y_t Y_{t-1}}{\sum_{t=2}^T Y_{t-1}^2}.$$

Is $\widetilde{\rho}_T$ a consistent estimator of ρ? Justify your answer.

(b) (8 pts.) Consider the following estimator from long regression

$$\begin{pmatrix} \widehat{\mu}_T \\ \widehat{\rho}_T \end{pmatrix} = \begin{pmatrix} \sum_{t=2}^T t^2 & \sum_{t=2}^T tY_{t-1} \\ \sum_{t=2}^T tY_{t-1} & \sum_{t=2}^T tY_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=2}^T tY_t \\ \sum_{t=2}^T Y_{t-1} Y_t \end{pmatrix}.$$

Is $(\widehat{\mu}_T, \widehat{\rho}_T)'$ a consistent estimator of $(\mu, \rho)'$? Justify your answer.

Some Useful Theorems and Lemmas

Theorem 1 *For any real number $|a| < 1$, there is $\sum_{s=1}^\infty s a^s = a(1-a)^{-2}$.*

Theorem 2 (Martingale Convergence Theorem) *Let $\{(X_t, \mathcal{F}_t)\}_{t \in \mathbb{Z}_+}$ be a martingale in L^2. If $\sup_t E\left[|X_t|^2\right] < \infty$, then $X_n \to X_\infty$ almost surely, where X_∞ is some element in L^2.*

Theorem 3 (Martingale CLT) *Let $\{X_{t,n}, \mathcal{F}_{t,n}\}$ be a martingale difference array such that $E[|X_{t,n}|^{2+\delta}] < \Delta < \infty$ for some $\delta > 0$ and for all t and n. If $\overline{\sigma}_n^2 > \delta_1 > 0$ for all n sufficiently large and $\frac{1}{n}\sum_{t=1}^n X_{t,n}^2 - \overline{\sigma}_n^2 \to_p 0$, then $n^{\frac{1}{2}} \overline{X}_n / \overline{\sigma}_n \to_d N(0,1)$.*

Theorem 4 (LLN of Linear Processes) *Suppose that Z_t is i.i.d. with mean zero and $E[|Z_0|] < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k|\varphi_k| < \infty$. Then $n^{-1}\sum_{t=1}^n X_t \to_{a.s.} 0$.*

Theorem 5 (CLT of Linear Processes) *Suppose that Z_t is i.i.d. with mean zero and $E[Z_0^2] = \sigma_Z^2 < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k^2 \varphi_k^2 < \infty$. Then $n^{-\frac{1}{2}}\sum_{t=1}^n X_t \to_d N\left[0, \varphi(1)^2 \sigma_Z^2\right]$.*

Theorem 6 (LLN of Sample Variance) *Suppose that Z_t is i.i.d. with mean zero and $E[Z_0^2] = \sigma_Z^2 < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k\varphi_k^2 < \infty$. Then*

$$\frac{1}{n}\sum_{t=1}^n X_t X_{t-h} \to_p \Gamma_X(h) = E\left[X_t X_{t-h}\right]. \tag{3}$$

Theorem 7 (Donsker) *Let $\{u_t\}$ be a sequence of random variables generated by $u_t = \sum_{k=0}^\infty \varphi_k \varepsilon_{t-k} = \varphi(L)\varepsilon_t$, where $\{\varepsilon_t\} \sim iid\ (0, \sigma_\varepsilon^2)$ with finite fourth moment and $\{\varphi_k\}$ is a sequence of constants with $\sum_{k=0}^\infty k|\varphi_k| < \infty$. Then $B_{u,n}(\cdot) = n^{-\frac{1}{2}}\sum_{t=1}^{[n\cdot]} u_t \to_d \lambda B(\cdot)$, where $\lambda = \sigma_\varepsilon \varphi(1)$.*