Department of Economics
UCLA

Spring 2014

Comprehensive Examination

Quantitative Methods

The exams consists of three part. You are required to answer all the questions in all the parts. Good luck!

Part I - 203A

1. The probability density of the random variable Y conditional on the random variable $X = x$ is known to be of the form

$$f_{Y|X=x}(y) = \begin{cases} \lambda(x) \;\; e^{-\lambda(x)\; y} & \text{if} \quad y > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda(x)$ is a function whose values are known to be nonnegative and bounded but it is otherwise unknown.

(a) Suppose that X is known to have a binomial distribution with unknown parameters (n, p). That is

$$\Pr(X = x) = \begin{pmatrix} n \\ x \end{pmatrix} \; p^x \; (1-p)^{n-x} \qquad x = 0, ..., n$$

Determine what can be identified from the distribution of (Y, X). Justify your answer.

(b) Suppose that X is continuously distributed on the interval $[1, 2]$ and it is known that for some positive, monotone increasing, and bounded, but otherwise unknown, function $m(x)$ and for some positive but otherwise unknown parameter α

$$\lambda(x) = \alpha + m(x)$$

Determine what can be identified from the distribution of (Y, X). Justify your answer. If a function or parameter is not identified, provide additional conditions under which it is identified and justify your answer.

(c) Suppose now that

$$Z = \frac{1}{E[Y|X = x]} - \eta$$

where η is a random variable distributed independently of X with an unknown strictly increasing distribution function F_η, and that Y and η are not observed. Suppose that X is continuously distributed with support R and that the only objects that are observed are the density of X and

$$\Pr(Z \geq 0|X = x)$$

for all x in the support of X. Determine what functions and parameters can be identified from these observed objects. If a function or parameter is not identified, provide additional conditions under which it is identified and justify your answer.

2. The random variable Y is determined by the values of random variables X, β, and ε according to the model

$$Y = \beta\, X + \varepsilon$$

where (X, β, ε) is distributed Normal with mean $(\mu_X, \mu_\beta, \mu_\varepsilon)$ and variance-covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_{XX} & \sigma_{X\beta} & \sigma_{X\varepsilon} \\ \sigma_{X\beta} & \sigma_{\beta\beta} & \sigma_{\beta\varepsilon} \\ \sigma_{X\varepsilon} & \sigma_{\beta\varepsilon} & \sigma_{\varepsilon\varepsilon} \end{bmatrix}$$

(a) Derive an expression in terms of the parameters, and as simple as possible, for the probability that $Y \geq 0$ conditional on $X = x$.

(b) Derive an expression in terms of the parameters, and as simple as possible, for the probability that $Y \geq 0$.

(c) Obtain an expression for the mean of Y.

(d) Obtain an expression for the variance of Y.

(e) Suppose that ε is known to be distributed independently of the vector (X, β). What can you say about the values of the mean and variance parameters of (X, β, ε)? In this case, are β and ε independent conditional on X? Explain.

(f) Suppose that X and β are independently distributed. Let A denote the intersection of the following three events: $(X \geq 1)$, $(\beta \geq 1)$, and $(\beta X \leq 2)$. Obtain an expression, in terms of the parameters, for the probability of A.

Note 1: You may use $\phi(u)$ and $\Phi(u)$ to denote, respectively, the density and cumulative distribution of a $N(0, 1)$ random variable.

Note 2: You might find useful to recall that the Moment Generating Function of a random vector W distributed $N(\mu, \Sigma)$ is $M_W(t) = e^{t'\mu + (1/2)t'\Sigma t}$.

Part II - 203B

1. *No derivation is required for the two short questions below; your derivation will not be read anyway.*

   (a) Suppose that
   $$y_i = x_i\beta + \varepsilon_i$$
   with $(y_i, x_i)'$ $i = 1, \ldots, n$ i.i.d., and $E[x_i\varepsilon_i] = 0$. We do NOT assume that $E[\varepsilon_i^2 | x_i]$ is constant. You are given the following data set with $n = 3$:
   $$\begin{bmatrix} y_1 & x_1 \\ y_2 & x_2 \\ y_3 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$$
   Using the asymptotic normality of the OLS estimator as well as White's heteroscedasticity corrected estimator of the asymptotic variance, produce the 95% confidence interval of β. If your answer involves a square root, try to simplify as much as you can.

   (b) We are given the following model:
   $$\begin{aligned} q^d &= \alpha_1 \cdot p + \alpha_2 \cdot y + \epsilon^d && \text{(Demand)} \\ q^s &= \beta_1 \cdot p + \beta_3 \cdot x + \epsilon^s && \text{(Supply)} \\ q &= q^d = q^s && \text{(Equilibrium)} \end{aligned}$$
   We assume that the random vector $(y, x)'$ is independent of the random vector $(\epsilon^d, \epsilon^s)'$. We also assume that $E[\epsilon^d] = E[\epsilon^s] = 0$. We have identified
   $$\begin{aligned} E[p | y, x] &= y - 2x \\ E[q | y, x] &= 3y + 5x \end{aligned}$$
   What are numerical values of β_1 and α_1?

2. *Your answer to this question will be evaluated based on the logical validity and coherence of your argument.*

   Suppose that $y_1, \ldots, y_n$ are independent and identically distributed scalar random variables. Their common distribution is $N(\theta, \sigma^2)$. Using the Gauss-Markov Theorem stated below, derive the best linear unbiased estimator of θ. (You do not need to prove the Gauss-Markov Theorem, but you should argue why this question satisfies the premises of the theorem. No other argument will be accepted.)

   Gauss-Markov Theorem: If (i) $y = X\beta + \varepsilon$; (ii) X is a nonstochastic matrix; (iii) X has a full column rank (Columns of X are linearly independent); (iv) $E[\varepsilon] = 0$; (iv) $E[\varepsilon\varepsilon'] = \sigma^2 I_n$ for some positive number σ^2; then OLS is BLUE.

3. *Your answer to this question will be evaluated based on the logical validity of your argument as well as the numerical accuracy.*

   Suppose that

$$y_i = x_i\beta + \varepsilon_i$$

   We observe $(y_i, x_i, z_i)'$, $i = 1, \ldots, n$ i.i.d. We assume that (1) the random vector $(x_i, z_i)'$ is independent of ε_i; (2) $E[\varepsilon_i] = 0$ and $E[\varepsilon_i^2] = 1$; (3) $E[x_i^2] = E[z_i^2] = 1$ and $E[x_i z_i] = \frac{1}{2}$. Derive the asymptotic distribution of

$$\begin{bmatrix} \sqrt{n}\left(\widehat{\beta}_{OLS} - \beta\right) \\ \sqrt{n}\left(\widehat{\beta}_{IV} - \beta\right) \end{bmatrix}$$

   where

$$\widehat{\beta}_{OLS} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}, \quad \widehat{\beta}_{IV} = \frac{\sum_{i=1}^{n} z_i y_i}{\sum_{i=1}^{n} z_i x_i}.$$

   Make sure that the asymptotic variance matrix consists of concrete numbers; if you stop with abstract formulae or your numbers are incorrect, your answer will be understood to be at best 30% complete.

Part III - 203C

1. Suppose that $\{X_t\}_t$ is a second order auto-regressive process, i.e.

$$X_t = \phi_o X_{t-1} + \frac{1}{2} X_{t-2} + u_t,$$

where $u_t \sim i.i.d.(0, \sigma_u^2)$, $\sigma_u^2 > 0$ and u_t has finite 4-th moment. We have n observations on X_t: $\{X_t\}_{t=1}^n$.

(a) Suppose $|\phi_o| < \frac{1}{2}$. Is $\{X_t\}_t$ a weakly stationary process? Justify your answer.

(b) Suppose that $\phi_o = 0$. Derive the auto-covariance function of $\{X_t\}_t$.

(c) Suppose that $\phi_o = 0$ and we know the value of ϕ_o. Derive the long-run variance (LRV) of $\{X_t\}_t$. Provide a LRV estimator which is root-n consistent.

(d) Suppose that $|\phi_o| < \frac{1}{2}$. Show that ϕ_o is identified by the following moment condition:

$$E\left[(X_t - \phi_o X_{t-1} - \frac{1}{2} X_{t-2}) X_{t-1}\right] = 0. \tag{1}$$

(e) Suppose that $\phi_o = 0$ and we do not know the value of ϕ_o. From the moment condition (1), we can construct the method of moment (MM) estimator of ϕ_o as

$$\widehat{\phi}_n = \frac{\sum_{t=3}^{n} (X_t - \frac{1}{2} X_{t-2}) X_{t-1}}{\sum_{t=2}^{n} X_t^2}.$$

Derive the asymptotic distribution of the above MM estimator.

(f) Suppose that $\phi_o = 0$ and we do not know the value of ϕ_o. Show that ϕ_o is identified by the moment conditions:

$$\begin{pmatrix} E\left[(X_t - \phi_o X_{t-1} - 2X_{t-2}) X_{t-1}\right] \\ E\left[(X_t - \phi_o X_{t-1} - 2X_{t-2}) X_{t-2}\right] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2}$$

Find the asymptotic variance of the optimally weighted GMM estimator of ϕ_o based on the moment conditions (2).

(g) Compare the asymptotic variances of the optimally weighted GMM estimator and the MM estimator studied in (e) and explain your finding.

2. Consider the following model

$$X_t = \alpha_o t + u_t \text{ with } u_t = \sqrt{t}\varepsilon_t \tag{3}$$

where $\varepsilon_t \sim i.i.d.(0, \sigma_\varepsilon^2)$ with $\sigma_\varepsilon^2 > 0$, ε_t has finite 4-th moment, and α_o and σ_ε^2 are unknown parameters. We have n observations on X_t: $\{X_t\}_{t=1}^n$.

(a) Derive the asymptotic distribution of the LS estimator of α_o:

$$\widehat{\alpha}_n = \frac{\sum_{t=1}^n tX_t}{\sum_{t=1}^n t^2}. \tag{4}$$

(b) Construct a consistent estimator $\widehat{\sigma}_{\varepsilon,n}^2$ of σ_ε^2. Derive the asymptotic distribution of your estimator.

(c) Using the estimator $\widehat{\sigma}_{\varepsilon,n}^2$ of σ_ε^2, one can construct an estimator of the variance of u_t as $\widehat{\sigma}_{u_t,n}^2 = t\widehat{\sigma}_{\varepsilon,n}^2$. Consider the generalized LS (GLS) estimator of α_o:

$$\widehat{\alpha}_{gls,n} = \frac{\sum_{t=1}^n tX_t\widehat{\sigma}_{u_t,n}^{-2}}{\sum_{t=1}^n t^2\widehat{\sigma}_{u_t,n}^{-2}}. \tag{5}$$

Derive the asymptotic distribution of $\widehat{\alpha}_{gls,n}$. Compare the asymptotic variances of the LS estimator and the GLS estimator and explain your findings.

Some Useful Theorems and Lemmas

Theorem 1 (Martingale Convergence Theorem) *Let* $\{(X_t, \mathcal{F}_t)\}_{t \in \mathbb{Z}_+}$ *be a martingale in* L^2. *If* $\sup_t E\left[|X_t|^2\right] < \infty$, *then* $X_n \to X_\infty$ *almost surely, where* X_∞ *is some element in* L^2.

Theorem 2 (Martingale CLT) *Let* $\{X_{t,n}, \mathcal{F}_{t,n}\}$ *be a martingale difference array such that* $E[|X_{t,n}|^{2+\delta}] < \Delta < \infty$ *for some* $\delta > 0$ *and for all* t *and* n. *If* $\overline{\sigma}_n^2 > \delta_1 > 0$ *for all* n *sufficiently large and* $\frac{1}{n}\sum_{t=1}^n X_{t,n}^2 - \overline{\sigma}_n^2 \to_p 0$, *then* $n^{\frac{1}{2}}\overline{X}_n / \overline{\sigma}_n \to_d N(0,1)$.

Theorem 3 (LLN of Sample Variance) *Suppose that* Z_t *is i.i.d. with mean zero and* $E[Z_0^2] = \sigma_Z^2 < \infty$. *Let* $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, *where* φ_k *is a sequence of real numbers with* $\sum_{k=0}^\infty k\varphi_k^2 < \infty$. *Then*

$$\frac{1}{n}\sum_{t=1}^n X_t X_{t-h} \to_p \Gamma_X(h) = E\left[X_t X_{t-h}\right]. \tag{6}$$

Theorem 4 (Donsker) *Let* $\{u_t\}$ *be a sequence of random variables generated by* $u_t = \sum_{k=0}^\infty \varphi_k \varepsilon_{t-k} = \varphi(L)\varepsilon_t$, *where* $\{\varepsilon_t\} \sim iid\ (0, \sigma_\varepsilon^2)$ *with finite fourth moment and* $\{\varphi_k\}$ *is a sequence of constants with* $\sum_{k=0}^\infty k\,|\varphi_k| < \infty$. *Then* $B_{u,n}(\cdot) = n^{-\frac{1}{2}}\sum_{t=1}^{[n\cdot]} u_t \to_d \lambda B(\cdot)$, *where* $\lambda = \sigma_\varepsilon \varphi(1)$.

Theorem 5 *For any natural number* n, *we have*

$$\sum_{k=1}^n k = \frac{n(n+1)}{2} \text{ and } \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$