First Year Quantitative Comp Exam
Spring, 2013

Instruction: There are three parts. Answer every question in every part.

Part I - 203A

Question I-1

Let $U_1, U_2, \ldots,$ be a sequence of i.i.d. random variables having the uniform distribution on $[0,1]$ and define

$$Y_n = \left(\prod_{i=1}^{n} U_i\right)^{-\frac{1}{n}}$$

Show that

$$\sqrt{n}\left(Y_n - \exp(1)\right) \xrightarrow{d} \mathcal{N}\left(0, \exp(1)^2\right)$$

Hint: The integration by parts formula may be useful

$$\int_{[a,b]} u(x)v'(x)dx = u(x)v(x)|_a^b - \int_{[a,b]} u'(x)v(x)dx$$

Question I-2

Consider a sample $\{X_i\}_{i=1}^n$ where X_i are independent random variables and X_i is distributed as bivariate normal with mean vector and covariance matrix given by

$$\boldsymbol{\mu}_i \equiv \begin{bmatrix} \mu_i \\ \mu_i \end{bmatrix} \qquad \boldsymbol{\Sigma} \equiv \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

where $\mu_i \in \mathbb{R}$ and $\sigma^2 > 0$. Find the MLE of $\{\mu_i\}_{i=1}^n$ and σ^2. Show that the MLE for σ^2 is inconsistent as $n \to \infty$

Question I-3

Missing data is pervasive in economics. This problem studies what can be identified in linear regression models in the presence of missing data.

Consider first the regression model

$$Y = \beta_0 + \beta_1 X + \beta_2 Z + \epsilon$$
$$\mathbb{E}\left[\epsilon | X, Z\right] = 0$$

If we observed an i.i.d. sample from the joint distribution (Y, X, Z), the vector $\beta \equiv (\beta_0, \beta_1, \beta_2)$ is identified and we can estimate it using OLS. Suppose now that we observe data on (Y, X, Z) but that some of the data on X (only) is missing.

One way to formalize this is the following: Let $\{Y_i, X_i, Z_i, M_i\}_{i=1}^{\infty}$ be an i.i.d. sequence from a distribution F. For each observation i we observe (Y, Z, M) and $X(1 - M)$. The variable M is a binary variable equal to 0 if X is observed and 1 otherwise. We assume that the data is missing at random in the sense that

$$\mathbb{E}\left[\epsilon | M, X, Z\right] = 0 \tag{1}$$

1. A common practice in empirical work (according to a recent estimate, about 20% of papers with missing data problems resort to this method) in such situations is the following: replace X by 0 whenever it is missing and add a dummy equal to 1 if the observation has a missing X. Formally, researchers run the OLS regression

   $$Y = \alpha_0 + \alpha_1 M + \alpha_2 (1 - M) X + \alpha_3 Z + u \tag{2}$$

   where $E[uM] = E[u(1 - M)X] = E[uZ] = E[u] = 0$. In this part we will see whether the OLS estimator for α_2 converges to β_1. To make some progress, we write

   $$X = \gamma_0 + \gamma_1 Z + v$$

   and assume that in addition to (1)

   $$\mathbb{E}\left[v | M, Z\right] = 0$$

   Show that the OLS estimator for α_2 will be consistent for β_1 if either $\gamma_1 = 0$ or $\beta_1 = 0$.

2. Show that if we add the regressor MZ to (2), then the OLS estimator for α_2 will be consistent for β_1.

Part II - 203B

Question II-1

Suppose that

$$y_i = x_i^3 + \varepsilon_i$$

such that $(x_i, \varepsilon_i)'$ $i = 1, 2, \ldots$ is i.i.d. and

$$\begin{pmatrix} x_i \\ \varepsilon_i \end{pmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

1. Let

   $$\widehat{\beta} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$

   denote the OLS estimator of β. What is the probability limit of $\widehat{\beta}$ as $n \to \infty$?

   From the model $y = x_i^3 + \varepsilon$, we can calculate the derivative $dy/\,dx = 3x_i^2$, based on which we can define the average derivative

   $$E\left[3x_i^2\right]$$

   Is the average derivative equal to the probability limit of $\widehat{\beta}$?

2. Suppose now that

   $$\begin{pmatrix} x_i \\ \varepsilon_i \end{pmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

   Does your conclusion in the previous question change or not?

   Hint: The moment generating function of $\mathcal{N}\left(\mu, \sigma^2\right)$ is $\exp\left(\mu t + \frac{\sigma^2}{2} t^2\right)$.

Question II-2

Consider the model

$$y_i = x_i \beta + \varepsilon_i$$
$$x_i = z_{i1}\pi_1 + z_{i2}\pi_2 + v_i$$

where we assume that (i) $(z_{i1}, z_{i2})'$ is independent of $(\varepsilon_i, v_i)'$; (ii) $(\varepsilon_i, v_i)'$ has a mean equal to zero; and (iii) z_{i1} and z_{i2} are independent of each other with both means equal to zero. We assume that $(x_i, z_{i1}, z_{i2}, \varepsilon_i, v_i)'$ is iid. Let

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^n z_{i1} y_i}{\sum_{i=1}^n z_{i1} x_i}, \quad \widehat{\beta}_2 = \frac{\sum_{i=1}^n z_{i2} y_i}{\sum_{i=1}^n z_{i2} x_i}$$

You may assume for simplicity that the second moments of $z_{i1}, z_{i2}, \varepsilon_i, v_i$ are all 1.

1. Derive the asymptotic distribution of

   $$\begin{bmatrix} \sqrt{n}\left(\widehat{\beta}_1 - \beta\right) \\ \sqrt{n}\left(\widehat{\beta}_2 - \beta\right) \end{bmatrix}$$

2. Assume that the above vector converges to $\mathcal{N}(0, \Sigma)$. Consider a class of estimators $t\widehat{\beta}_1 + (1-t)\widehat{\beta}_2$ indexed by t. Find t that minimizes the asymptotic variance of $\sqrt{n}\left(t\widehat{\beta}_1 + (1-t)\widehat{\beta}_2 - \beta\right)$. What is the minimized asymptotic variance?

3. Derive the asymptotic distribution of 2SLS for this model. (You do not need to show how to derive the asymptotic distribution of 2SLS for the general case. You are allowed to state the asymptotic variance formula of 2SLS, and then use specific values from this question.) Compare the asymptotic variance of 2SLS with the minimized variance derived in (2) above.

Question II-3

Suppose that $Z_1, \ldots, Z_n$ are i.i.d., and their common PDF is given by $f(z; \theta_1, \theta_2)$. Both θ_1 and θ_2 are scalars. Let

$$s_1 = \frac{\partial \log f(Z; \theta_1, \theta_2)}{\partial \theta_1}$$
$$s_2 = \frac{\partial \log f(Z; \theta_1, \theta_2)}{\partial \theta_2}$$

and let

$$\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} = E\begin{bmatrix} s_1^2 & s_1 s_2 \\ s_1 s_2 & s_2^2 \end{bmatrix}$$

Assume that $|\rho| < 1$. Here, the Z denotes a random variable with distribution identical to that of Z_i.

We consider two estimators of θ_1. The first estimator $\widehat{\theta}_1$ is a feasible estimator that solves

$$\max_{c_1, c_2} \sum_{i=1}^{n} \log f(Z_i; c_1, c_2)$$

The second estimator $\widetilde{\theta}_1$ is an infeasible estimator that solves

$$\max_{c_1} \sum_{i=1}^{n} \log f(Z_i; c_1, \theta_2)$$

i.e., it is based on the assumption that θ_2 is known. Prove that the asymptotic variance of $\sqrt{n}\left(\widetilde{\theta}_1 - \theta_1\right)$ is smaller than or equal to the asymptotic variance of $\sqrt{n}\left(\widehat{\theta}_1 - \theta_1\right)$. When are they equal to each other?

Part III - 203C

Question III-1

Let $X_1, ..., X_n$ be a random sample from a location-exponential family with density function

$$f(x;\theta) = \begin{cases} \exp(\theta - x), & \text{if } x \geq \theta \\ 0, & \text{otherwise} \end{cases},$$

where θ can be any finite constant, i.e. $\theta \in (-\infty, \infty)$. We are interested in testing

$$H_0 : \theta \leq \theta_o \text{ versus } H_1 : \theta > \theta_o.$$

1. Find the maximum likelihood estimator of θ.
2. Find likelihood ratio test with size $\alpha \in (0, 1)$.(note: you are supposed to find the explicit critical value and hence the explicit critical region.)
3. Find the power function of the likelihood ratio test in question (2).

Question III-2

Suppose that $\{Y_t\}$ is an auto-regressive process, i.e.

$$Y_t = \rho_o Y_{t-1} + u_t,$$

where $|\rho_o| < 1$ and $u_t \sim i.i.d.(0, \sigma_u^2)$. Suppose that Y_t is only observable at the even period, i.e. when t is an even number. and there exists a sample of observation for Y_t, i.e. $\{Y_{2t}\}_{t=1}^n$.

1. Given the sample $\{Y_{2t}\}_{t=1}^n$, we can define an estimator as

$$\widehat{\theta}_n = \frac{\sum_{t=2}^n Y_{2t} Y_{2t-2}}{\sum_{t=1}^n Y_{2t}^2}. \tag{3}$$

   Is $\widehat{\theta}_n$ a consistent estimator for ρ_o? Show the consistency of $\widehat{\theta}_n$ if your answer is "yes". Otherwise, construct a consistent estimator for ρ_o.

2. Find the asymptotic distribution of the consistent estimator you find in question (1).

Question III-3

Suppose that $\{e_t\}$ is generated from the following equation

$$e_t = \gamma_o + e_{t-1} + u_t,$$
$$u_t = \rho_o u_{t-1} + v_t,$$

where $e_0 = 0$, ρ_o and γ_o are some finite constants and $v_t \sim i.i.d.(0, \sigma_v^2)$.

1. Suppose that γ_o and ρ_o are known and you have data $\{e_t\}_{t=0}^n$ where $n \geq 2$. What is the optimal prediction of e_{n+1}? What is the related mean square prediction error?

2. Given the data $\{e_t\}_{t=1}^n$, one could estimate γ_o and ρ_o by

$$\widehat{\gamma}_n = \frac{1}{n-1}\sum_{t=1}^{n-1}(e_{t+1} - e_t) \text{ and } \widehat{\rho}_n = \frac{\sum_{t=1}^{n-1}\widehat{u}_{t+1}\widehat{u}_t}{\sum_{t=1}^{n-1}\widehat{u}_t^2} \tag{4}$$

where $\widehat{u}_t = e_t - e_{t-1} - \widehat{\gamma}_n$. If $|\rho_o| < 1$, are the estimators $\widehat{\gamma}_n$ and $\widehat{\rho}_n$ defined in equation (4) consistent? Justify your answer.

3. If $\rho_o \geq 1$, are the estimators $\widehat{\gamma}_n$ and $\widehat{\rho}_n$ defined in equation (4) still consistent? Justify your answer.

Some Useful Theorems and Lemmas

Theorem 1 (Martingale Convergence Theorem) *Let $\{(X_t, \mathcal{F}_t)\}_{t\in\mathbb{Z}_+}$ be a martingale in L^2. If $\sup_t E\left[|X_t|^2\right] < \infty$, then $X_n \to X_\infty$ almost surely, where X_∞ is some element in L^2.*

Theorem 2 (Martingale CLT) *Let $\{X_{t,n}, \mathcal{F}_{t,n}\}$ be a martingale difference array such that $E[|X_{t,n}|^{2+\delta}] < \Delta < \infty$ for some $\delta > 0$ and for all t and n. If $\overline{\sigma}_n^2 > \delta_1 > 0$ for all n sufficiently large and $\frac{1}{n}\sum_{t=1}^n X_{t,n}^2 - \overline{\sigma}_n^2 \to_p 0$, then $n^{\frac{1}{2}}\overline{X}_n/\overline{\sigma}_n \to_d N(0,1)$.*

Theorem 3 (LLN of Sample Variance) *Suppose that Z_t is i.i.d. with mean zero and $E[Z_0^2] = \sigma_Z^2 < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k\varphi_k^2 < \infty$. Then*

$$\frac{1}{n}\sum_{t=1}^n X_t X_{t-h} \to_p \Gamma_X(h) = E\left[X_t X_{t-h}\right]. \tag{5}$$

Theorem 4 (Donsker) *Let $\{u_t\}$ be a sequence of random variables generated by $u_t = \sum_{k=0}^\infty \varphi_k \varepsilon_{t-k} = \varphi(L)\varepsilon_t$, where $\{\varepsilon_t\} \sim iid\ (0, \sigma_\varepsilon^2)$ with finite fourth moment and $\{\varphi_k\}$ is a sequence of constants with $\sum_{k=0}^\infty k\,|\varphi_k| < \infty$. Then $B_{u,n}(\cdot) = n^{-\frac{1}{2}}\sum_{t=1}^{[n\cdot]} u_t \to_d \lambda B(\cdot)$, where $\lambda = \sigma_\varepsilon\varphi(1)$.*