Department of Economics
UCLA
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# Quantitative Methods Comprehensive Examination

Instructions: Answer **ALL questions** in Parts I, II, and III. Use a separate answer book for each Part. You have four hours to complete the exam.

## PART I (Based on ECON203A)

1. (10 pt.) Let $X$ denote a random vector with p.d.f. $f(x;\theta) = L(\theta;x)$. Assume that $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$. Let

$$C \equiv \left\{ x : \frac{L(\theta_1;x)}{L(\theta_0;x)} \geq k \right\},$$

where $k$ is chosen in such a way that

$$P[X \in C; H_0] = \alpha.$$

Prove that $C$ is the best critical region of size $\alpha$ for testing $H_0$ against the alternative $H_1$.

2. (10 pt.) Given a random vector $X$ with pdf $f(x;\theta)$, consider an unbiased estimator $Y = u(X)$ of $\theta$, where $\theta$ is a scalar. Suppose that $E[Y] = \theta$. Prove that we have

$$\operatorname{Var}(Y) \geq -\frac{1}{E\left[\frac{\partial^2 \log f(X;\theta)}{\partial \theta^2}\right]}$$

3. (10 pt.) Suppose that we have

$$\underset{n\times 1}{y} = \underset{n\times k}{X}\,\underset{k\times 1}{\beta} + \underset{n\times 1}{\epsilon}$$

where $X$ is nonstochastic with full column rank, $E[\epsilon] = 0$ and $E[\epsilon\epsilon'] = \sigma^2 I_n$. Prove that

$$\widehat{\beta} = (X'X)^{-1} X'y$$

has the smallest possible variance matrix among all the estimators of $\beta$ that takes the form $c = Cy$ such that $E[c] = \beta$.

4. (10 pt.) Suppose that $U \sim U(0,1)$. Given an arbitrary cdf $F$, prove that $Y \equiv F^{-1}(U)$ has a cdf equal to $F$.

5. (10 pt.) Suppose that $X_i$ are i.i.d. such that $E[\exp(tX_i)] = \exp\left(\frac{t^2}{2}\right)$. Derive the asymptotic distribution of

$$\sqrt{n}\left(n^{-1}\sum_{i=1}^{n} X_i^4\right)^2 \times \left(n^{-1}\sum_{i=1}^{n} X_i\right)$$

## PART II (Based on ECON203B)

1. Suppose that you have $n$ independent observations on $(x_i, y_i)$ from the model

$$y_i = x_i\beta + \varepsilon_i \qquad i = 1, ..., n$$

where $\varepsilon_i \sim N(0, \exp(w_i\gamma))$ with $w_i$ a $q-$dimensional subvector of $x_i$.

   (a) Derive the GLS estimator of $\beta$ assuming $\gamma$ is known.

   (b) Derive the ML estimator of $\beta$ assuming $\gamma$ is known. How is it related to the infeasible GLS estimator?

   (c) State conditions under which the infeasible GLS estimator is consistent and derive its asymptotic distribution.

   (d) Describe the feasible GLS approach when $\gamma$ is unknown.

   (e) Describe the ML estimator of $\beta$ assuming $\gamma$ is unknown. How is it related to the feasible GLS estimator?

2. Suppose that we collect Social Security data in order to analyze the effect of different demographic characteristics on individuals' earnings. The latter however are top-coded at \$100,000, i.e. earnings that are above \$100,000 are reported as 100,000. Set up an econometric model that accounts for top-coding. Suggest three estimators that estimate the model and discuss their relative merits. How would you estimate the marginal effect of a specific characteristic and how would you construct standard errors for your estimate?

## PART III (Based on ECON203C)

1. Consider the neo-classical regression model

$$y_i = x_i'\beta + w_i^*\gamma + u_i \qquad (i = 1, \ldots, n)$$

where $\beta$ is a $k \times 1$ vector of parameters, and $\gamma$ is a $p \times 1$ vector of parameters. Suppose that

$$E[x_i u_i] = 0,$$
$$w_i = w_i^* + v_i,$$

with

$$E[v_i u_i] = 0.$$

Suppose also that there exists a $p \times 1$ vector $z_i$ such that

$$z_i = g(w_i^*) + \varepsilon_i,$$

for some known $p \times 1$ vector valued function $g(\cdot)$.

(a) Demonstrate whether or not the vector $\beta$ be consistently estimated by a least-squares regression.

(b) Propose an instrumental variable estimator for $\beta$ using $z_i$ as a vector of instruments. Provide all the assumptions needed for $z_i$ to be a valid vector of instruments. Justify your answer as precisely as possible.

(c) Under the condition in (b), consider the following estimation procedure: (i) Estimate $\beta$ from a regression of $y$ on $X$; and (ii) Compute $\hat{y} = My$ (where $M = I - X(X'X)^{-1}X'$) and estimate $\gamma$ by computing the instrumental variable estimator from a regression of $\hat{y}$ on $w$, using $z$ as the instrumental variable for $w$.

(d) Compare the estimators for $\gamma$ from (b) and (c). Explain the difference and/or the similarity.

2. Consider the non-linear model given by

$$y_i = g\left(x_i'\beta_0\right) + \varepsilon_i,$$

for $i = 1, ..., n$. Define the following moment equations:

$$\begin{aligned}
\varphi_1\left(y_i, x_i, \beta\right) &= \left(y_i - g\left(x_i'\beta\right)\right)\frac{\partial g\left(x_i'\beta\right)}{\partial\beta} \\
\varphi_2\left(y_i, x_i, \beta\right) &= \left(y_i - g\left(x_i'\beta\right)\right)h\left(x_i\right), \\
\varphi_3\left(y_i, x_i, \beta\right) &= \left(y_i - g\left(x_i'\beta\right)\right)\left(\frac{\partial g\left(x_i'\beta\right)}{\partial\beta}\right)^2,
\end{aligned}$$

where

$$h\left(x_i\right) = \begin{pmatrix} \frac{\partial g\left(x_i'\beta\right)}{\partial\beta} \\ \left(\frac{\partial g\left(x_i'\beta\right)}{\partial\beta}\right)^2 \end{pmatrix},$$

and the notation $\lambda^2$ for a vector $\lambda = \left(\lambda_1, ..., \lambda_p\right)'$ is simply $\lambda = \left(\lambda_1^2, ..., \lambda_p^2\right)'$.

(a) What does one need to assume about $\varepsilon_i$ for $g\left(x_i'\beta_0\right)$ to be the true conditional mean of $y_i$, conditional on $x_i$.

(b) Assume now that the conditions in (a) are satisfied. Show that

$$\begin{aligned}
E\left[\varphi_1\left(y_i, x_i, \beta_0\right)\right] &= 0, \\
E\left[\varphi_2\left(y_i, x_i, \beta_0\right)\right] &= 0, \quad \text{and} \\
E\left[\varphi_3\left(y_i, x_i, \beta_0\right)\right] &= 0.
\end{aligned}$$

(c) Propose optimal GMM estimators for $\beta_0$ based on $\varphi_1\left(y_i, x_i, \beta\right)$ and $\varphi_2\left(y_i, x_i, \beta\right)$. Denote the estimators by $\widehat{\beta}_1$ and $\widehat{\beta}_2$, respectively.

(d) Provide the asymptotic properties for the estimators $\widehat{\beta}_1$ and $\widehat{\beta}_2$ defined in (b).

(e) Which of the two estimators would you prefer, $\widehat{\beta}_1$ or $\widehat{\beta}_2$? Justify your answer.

(f) Consider now a GMM estimator based on $\varphi_3\left(y_i, x_i, \beta\right)$, and denote this estimator by $\widehat{\beta}_3$. Which estimator would you prefer, $\widehat{\beta}_2$ or $\widehat{\beta}_3$? Justify your answer.