

### Spring 2000 Quantitative Methods Comprehensive Examination

There are three sections with two questions in each section. Please answer all the questions; use a separate blue book for each of the three sections. A set of tables for the normal and chi-square distributions are given at the end of the exam.

#### SECTION I. (Use separate blue book)

1. Let  $X$  have a binomial  $B(1, 1/2)$  distribution with  $Pr(X = 1) = 1/2$ . Conditional on  $X = x$ , the random variable  $Y$  has a Poisson distribution with parameter  $\lambda(1 + x)$ :

$$Pr(Y = y|X = x) = \frac{(\lambda(1 + x))^y \exp(-\lambda(1 + x))}{y!},$$

for  $x = 0, 1$ , and  $y = 0, 1, \dots$

- Calculate  $Pr(Y = 2)$ .
- Calculate  $Pr(X = 1|Y = 2)$ .
- Find an unbiased estimator for  $\lambda$  that is a function of  $Y$  alone.
- Find the variance for this estimator.
- Calculate the maximum likelihood estimator for  $\lambda$  (given the pair  $(X, Y)$ ).
- Calculate the variance for the maximum likelihood estimator in (e) conditional on  $X$  and compare it to the (unconditional) variance in (d).

2. Let the distribution of  $X$  have probability density function

$$f_X(x; \lambda) = \frac{1}{2} \lambda \exp(-|x|\lambda),$$

for  $-\infty < x < \infty$ . Let  $x_1, \dots, x_N$  be a random sample from this distribution.

- Calculate the mean and variance of  $X$ .
- Find a one-dimensional sufficient statistic for  $\lambda$ .
- Let  $N = 20$ ,  $\sum_{i=1}^N x_i = 20$ ,  $\sum_{i=1}^N |x_i| = 100$ ,  $\sum_{i=1}^N x_i^2 = 200$ . Find the maximum likelihood estimator. (d) Test the null hypothesis

$$H_0 : \quad \lambda = 0.125,$$

against the alternative hypothesis

$$H_1 : \quad \lambda \neq 0.125,$$

at the 5% level using a likelihood ratio test.

(e) Test the same null hypothesis using a Lagrange multiplier (score) test.

(f) Test the same null hypothesis at the 10% level using a Wald test.

**SECTION II. (Use separate blue book)**

1. Suppose that

$$y_i = x_i^* \beta + \varepsilon_i$$

$$x_i = x_i^* \cdot \nu_i$$

$$z_i = x_i^* \cdot \eta_i$$

where  $x_i^*$ ,  $\varepsilon_i$ ,  $\nu_i$  and  $\eta_i$  are independent random variables and  $(x_i^*, \varepsilon_i, \nu_i, \eta_i)$  is independent and identically distributed over  $i$ .

You observe  $(y_i, x_i, z_i)$  for  $i = 1, 2, \dots, n$ , and you are interested in estimating  $\beta$ .

Answer the following questions under the assumption that all relevant moments exist.

(a) Under what conditions is the OLS estimator

$$\hat{\beta}_{\text{OLS}} = \frac{\sum_{i=1}^n y_i x_i}{\sum_{i=1}^n x_i^2}$$

consistent for  $\beta$ ?

For the remaining questions assume that  $E[\nu_i] = E[\eta_i] = 1$ .

(b) Let  $\alpha$  be a fixed real number between 0 and 1. Under what conditions is

$$\tilde{\beta} = \frac{\sum_{i=1}^n \alpha y_i x_i + (1 - \alpha) y_i z_i}{\sum_{i=1}^n x_i z_i}$$

consistent?

(c) Derive the limiting distribution of  $\sqrt{n}(\tilde{\beta} - \beta)$ , where  $\tilde{\beta}$  is defined in (b).

(d) What is the best choice of  $\alpha$ ?

In parts (c) and (d) assume that the conditions of part (b) hold so that that  $\tilde{\beta}$  is consistent.

2. Suppose that it is known that

$$\sqrt{n} \left( \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} - \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right) \xrightarrow{d} N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma \right)$$

where  $\Sigma$  is unknown, but can be consistently estimated by  $\hat{\Sigma}$ .

Using these estimators and a sample of size  $n = 100$ , you estimate  $\beta_1$  and  $\beta_2$  to be 0 and  $-1$ , respectively. Your estimate of  $\Sigma$  is  $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ .

(a) Construct an asymptotically valid 95% confidence interval for  $\exp(\beta_1)$ .

(b) Construct an asymptotically valid 95% confidence interval for  $\beta_1\beta_2$ .

(c) Justify your methodology of parts (a) and (b).

**SECTION III. (Use separate blue book)**

1. Suppose that for  $i = 1, \dots, n$ ,

$$\begin{pmatrix} \alpha_i \\ x_{i1} \\ x_{i2} \\ u_{i1} \\ u_{i2} \end{pmatrix} \stackrel{\text{i.i.d.}}{\sim} N(0, \Sigma),$$

where

$$\Sigma = \begin{pmatrix} \sigma_\alpha^2 & \sigma_{\alpha x} & \sigma_{\alpha x} & 0 & 0 \\ \sigma_{\alpha x} & \sigma_x^2 & 0 & 0 & 0 \\ \sigma_{\alpha x} & 0 & \sigma_x^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_u^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_u^2 \end{pmatrix}.$$

Suppose also that for  $i = 1, \dots, n$  and  $t = 1, 2$ ,

$$y_{it} = \alpha_i + \beta x_{it} + u_{it}.$$

(a) Let  $b$  denote the OLS coefficient in a regression of  $y_{it}$  on  $x_{it}$  (without a constant). Derive the probability limit of  $b$  as  $n \rightarrow \infty$ .

(b) Let  $b_w$  denote the OLS coefficient in a regression of  $y_{it} - \bar{y}_i$  on  $x_{it} - \bar{x}_i$ , where  $\bar{y}_i = \frac{1}{2}(y_{i1} + y_{i2})$ , and  $\bar{x}_i = \frac{1}{2}(x_{i1} + x_{i2})$ . Let  $b_d$  denote the OLS coefficient in a regression of  $(y_{i2} - y_{i1})$  on  $(x_{i2} - x_{i1})$ . Show that

$$b_w = b_d.$$

(c) Derive the probability limit of  $b_d$  as  $n \rightarrow \infty$ .

(d) Derive the limiting distribution (as  $n \rightarrow \infty$ ) of

$$\sqrt{n}(b_d - \beta).$$

Try to obtain as simple an expression as possible based on the given assumptions.

(e) Suppose that instead of  $x_{it}$  you observe

$$w_{it} = x_{it} + \epsilon_{it}$$

where  $\epsilon_{it}$  is  $N(0, \sigma_\epsilon^2)$ , independent across both  $i$  and  $t$ , and independent of all other variables. Derive the probability limit of the within estimator that uses  $w_{it}$  in place of  $x_{it}$ .

2. We are interested in estimating parameters of a production function for price-taking, competitive firms. For  $i = 1, \dots, n$ , let  $y_i$  denote the output,  $l_i$  the labor input, and  $k_i$  the capital input, for firm  $i$ . We also observe  $p_{li}$ , the wage rate faced by firm  $i$ , and  $p_{ki}$ , the cost of capital for firm  $i$ . (All variables are in logs). Suppose that the following holds:

$$\begin{aligned} y_i &= \gamma_0 + \gamma_1 l_i + \gamma_2 k_i + v_i \\ l_i &= \alpha_0 + \alpha_1 p_{li} + \alpha_2 p_{ki} + u_{i1} \\ k_i &= \beta_0 + \beta_1 p_{li} + \beta_2 p_{ki} + u_{i2} \end{aligned}$$

Assume that the prices are “fixed” and that

$$\begin{pmatrix} v_i \\ u_{i1} \\ u_{i2} \end{pmatrix} \stackrel{\text{i.i.d.}}{\sim} N(0, \Sigma).$$

- (a) Derive the reduced form associated with this system of equations.
- (b) Describe the Indirect Least Squares (ILS) estimator for the structural parameters. In particular, obtain expressions for the estimates of  $\gamma_1$  and  $\gamma_2$ .
- (c) Explain the indirect generalized least squares (IGLS) approach for a general SEM, and comment on whether there would be an efficiency gain in the particular case given here.
- (d) Describe the 2SLS estimator and explain how it relates to the ILS estimator described in part (b).

Table A.2 Chi-square cumulative distribution function.

k	G <sub>1</sub> (·)										
	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
1	0.00	0.02	0.04	0.06	0.10	0.15	0.21	0.27	0.36	0.45	0.57
2	0.10	0.21	0.33	0.45	0.58	0.71	0.86	1.02	1.20	1.39	1.60
3	0.35	0.58	0.80	1.01	1.21	1.42	1.64	1.87	2.11	2.37	2.64
4	0.71	1.06	1.37	1.65	1.92	2.19	2.47	2.75	3.05	3.36	3.69
5	1.15	1.61	1.99	2.34	2.67	3.00	3.33	3.66	4.00	4.35	4.71
6	1.64	2.20	2.66	3.07	3.45	3.83	4.20	4.57	4.95	5.35	5.77
7	2.17	2.83	3.36	3.82	4.25	4.67	5.08	5.49	5.91	6.35	6.80
8	2.73	3.49	4.08	4.59	5.07	5.53	5.98	6.42	6.88	7.34	7.81
9	3.33	4.17	4.82	5.38	5.90	6.39	6.88	7.36	7.84	8.34	8.86
10	3.94	4.87	5.57	6.18	6.74	7.27	7.78	8.30	8.81	9.34	9.89
11	4.57	5.58	6.34	6.99	7.58	8.15	8.70	9.24	9.78	10.34	10.92
12	5.23	6.30	7.11	7.81	8.44	9.03	9.61	10.18	10.76	11.34	11.95
13	5.89	7.04	7.90	8.63	9.30	9.93	10.53	11.13	11.73	12.34	12.97
14	6.57	7.79	8.70	9.47	10.17	10.82	11.45	12.08	12.70	13.34	14.00
15	7.26	8.55	9.50	10.31	11.04	11.72	12.38	13.03	13.68	14.34	15.02
16	7.96	9.31	10.31	11.15	11.91	12.62	13.31	13.98	14.66	15.34	16.04
17	8.67	10.09	11.12	12.00	12.79	13.53	14.24	14.94	15.63	16.34	17.06
18	9.39	10.86	11.95	12.86	13.68	14.44	15.17	15.89	16.61	17.34	18.09
19	10.12	11.65	12.77	13.72	14.56	15.35	16.11	16.85	17.59	18.34	19.11
20	10.85	12.44	13.60	14.58	15.45	16.27	17.05	17.81	18.57	19.34	20.13
25	14.61	16.47	17.82	18.94	19.94	20.87	21.75	22.62	23.47	24.34	25.22
30	18.49	20.60	22.11	23.36	24.48	25.51	26.49	27.44	28.39	29.34	30.31
35	22.47	24.80	26.46	27.84	29.05	30.18	31.25	32.28	33.31	34.34	35.39
40	26.51	29.05	30.86	32.34	33.66	34.87	36.02	37.13	38.23	39.34	40.46
45	30.61	33.35	35.29	36.88	38.29	39.58	40.81	42.00	43.16	44.34	45.53
50	34.76	37.69	39.75	41.45	42.94	44.31	45.61	46.86	48.10	49.33	50.59
55	38.96	42.06	44.24	46.04	47.61	49.06	50.42	51.74	53.04	54.33	55.65
60	43.19	46.46	48.76	50.64	52.29	53.81	55.24	56.62	57.98	59.33	60.71
65	47.45	50.88	53.29	55.26	56.99	58.57	60.07	61.51	62.92	64.33	65.77
70	51.74	55.33	57.84	59.90	61.70	63.35	64.90	66.40	67.87	69.33	70.82
75	56.05	59.79	62.41	64.55	66.42	68.13	69.74	71.29	72.81	74.33	75.88
80	60.39	64.28	66.99	69.21	71.14	72.92	74.58	76.19	77.76	79.33	80.93
85	64.75	68.78	71.59	73.88	75.88	77.71	79.43	81.09	82.71	84.33	85.98
90	69.13	73.29	76.20	78.56	80.62	82.51	84.29	85.99	87.67	89.33	91.02
95	73.52	77.82	80.81	83.25	85.38	87.32	89.14	90.90	92.62	94.33	96.07
100	77.93	82.36	85.44	87.95	90.13	92.13	94.00	95.81	97.57	99.33	101.11

Example: If  $W \sim \chi^2(6)$ , then  $\Pr(W \leq 4.20) = G_6(4.20) = 0.35$ .

Table A.2 (continued)

k	G <sub>1</sub> (·)										
	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	0.975	0.990	0.995
1	0.71	0.87	1.07	1.32	1.64	2.07	2.71	3.84	5.02	6.63	7.88
2	1.83	2.10	2.41	2.77	3.22	3.79	4.61	5.99	7.38	9.21	10.60
3	2.95	3.28	3.66	4.11	4.64	5.32	6.25	7.81	9.35	11.34	12.84
4	4.04	4.44	4.88	5.39	5.99	6.74	7.78	9.49	11.14	13.28	14.86
5	5.13	5.57	6.06	6.63	7.29	8.12	9.24	11.07	12.83	15.09	16.75
6	6.21	6.69	7.23	7.84	8.56	9.45	10.64	12.59	14.45	16.81	18.55
7	7.28	7.81	8.38	9.04	9.80	10.75	12.02	14.07	16.01	18.48	20.28
8	8.35	8.91	9.52	10.22	11.03	12.03	13.36	15.51	17.53	20.09	21.95
9	9.41	10.01	10.66	11.39	12.24	13.29	14.68	16.92	19.02	21.67	23.59
10	10.47	11.10	11.78	12.55	13.44	14.53	15.99	18.31	20.48	23.21	25.19
11	11.53	12.18	12.90	13.70	14.63	15.77	17.28	19.68	21.92	24.72	26.76
12	12.58	13.27	14.01	14.85	15.81	16.99	18.55	21.03	23.34	26.22	28.30
13	13.64	14.35	15.12	15.98	16.98	18.20	19.81	22.36	24.74	27.69	29.82
14	14.69	15.42	16.22	17.12	18.15	19.41	21.06	23.68	26.12	29.14	31.32
15	15.73	16.49	17.32	18.25	19.31	20.60	22.31	25.00	27.49	30.58	32.80
16	16.78	17.56	18.42	19.37	20.47	21.79	23.54	26.30	28.85	32.00	34.27
17	17.82	18.63	19.51	20.49	21.61	22.98	24.77	27.59	30.19	33.41	35.72
18	18.87	19.70	20.60	21.60	22.76	24.16	25.99	28.87	31.53	34.31	37.16
19	19.91	20.76	21.69	22.72	23.90	25.33	27.20	30.14	32.85	36.19	38.58
20	20.95	21.83	22.77	23.83	25.04	26.50	28.41	31.41	34.17	37.57	40.00
25	26.14	27.12	28.17	29.34	30.68	32.28	34.38	37.65	40.65	44.31	46.93
30	31.32	32.38	33.53	34.80	36.25	37.99	40.26	43.77	46.98	50.89	53.67
35	36.47	37.62	38.86	40.22	41.78	43.64	46.06	49.80	53.20	57.94	60.27
40	41.62	42.85	44.16	45.62	47.27	49.24	51.81	55.76	59.34	63.69	66.77
45	46.76	48.06	49.45	50.98	52.73	54.81	57.51	61.66	65.41	69.96	73.17
50	51.89	53.26	54.72	56.33	58.16	60.35	63.17	67.50	71.42	76.15	79.49
55	57.02	58.45	59.98	61.66	63.58	65.86	68.80	73.31	77.38	82.29	85.75
60	62.13	63.63	65.23	66.98	68.97	71.34	74.40	79.08	83.30	88.38	91.95
65	67.25	68.80	70.46	72.28	74.35	76.81	79.97	84.82	89.18	94.42	98.11
70	72.36	73.97	75.69	77.58	79.71	82.26	85.53	90.53	95.02	100.43	104.21
75	77.46	79.13	80.91	82.86	85.07	87.69	91.06	96.22	100.84	106.39	110.29
80	82.57	84.28	86.12	88.13	90.41	93.11	96.58	101.88	106.63	112.33	116.32
85	87.67	89.43	91.32	93.39	95.73	98.51	102.08	107.52	112.39	118.24	122.32
90	92.76	94.58	96.52	98.65	101.05	103.90	107.57	113.15	118.14	124.12	128.30
95	97.85	99.72	101.72	103.90	106.36	109.29	113.04	118.75	123.86	129.97	134.25
100	102.95	104.86	106.91	109.14	111.67	114.66	118.50	124.34	129.56	135.81	140.17

# Appendix A

## Statistical and Data Tables

Table A.1 Standard normal cumulative distribution function.

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.500	0.504	0.508	0.512	0.516	0.520	0.524	0.528	0.532	0.536
0.10	0.540	0.544	0.548	0.552	0.556	0.560	0.564	0.567	0.571	0.575
0.20	0.579	0.583	0.587	0.591	0.595	0.599	0.603	0.606	0.610	0.614
0.30	0.618	0.622	0.626	0.629	0.633	0.637	0.641	0.644	0.648	0.652
0.40	0.655	0.659	0.663	0.666	0.670	0.674	0.677	0.681	0.684	0.688
0.50	0.691	0.695	0.698	0.702	0.705	0.709	0.712	0.716	0.719	0.722
0.60	0.726	0.729	0.732	0.736	0.739	0.742	0.745	0.749	0.752	0.755
0.70	0.758	0.761	0.764	0.767	0.770	0.773	0.776	0.779	0.782	0.785
0.80	0.788	0.791	0.794	0.797	0.800	0.802	0.805	0.808	0.811	0.813
0.90	0.816	0.819	0.821	0.824	0.826	0.829	0.831	0.834	0.836	0.839
1.00	0.841	0.844	0.846	0.848	0.851	0.853	0.855	0.858	0.860	0.862
1.10	0.864	0.867	0.869	0.871	0.873	0.875	0.877	0.879	0.881	0.883
1.20	0.885	0.887	0.889	0.891	0.893	0.894	0.896	0.898	0.900	0.901
1.30	0.903	0.905	0.907	0.908	0.910	0.911	0.913	0.915	0.916	0.918
1.40	0.919	0.921	0.922	0.924	0.925	0.926	0.928	0.929	0.931	0.932
1.50	0.933	0.934	0.936	0.937	0.938	0.939	0.941	0.942	0.943	0.944
1.60	0.945	0.946	0.947	0.948	0.949	0.951	0.952	0.953	0.954	0.954
1.70	0.955	0.956	0.957	0.958	0.959	0.960	0.961	0.962	0.962	0.963
1.80	0.964	0.965	0.966	0.966	0.967	0.968	0.969	0.969	0.970	0.971
1.90	0.971	0.972	0.973	0.973	0.974	0.974	0.975	0.976	0.976	0.977
2.00	0.977	0.978	0.978	0.979	0.979	0.980	0.980	0.981	0.981	0.982
2.10	0.982	0.983	0.983	0.983	0.984	0.984	0.985	0.985	0.985	0.986
2.20	0.986	0.986	0.987	0.987	0.987	0.988	0.988	0.988	0.989	0.989
2.30	0.989	0.990	0.990	0.990	0.990	0.991	0.991	0.991	0.991	0.992
2.40	0.992	0.992	0.992	0.992	0.993	0.993	0.993	0.993	0.993	0.994
2.50	0.994	0.994	0.994	0.994	0.994	0.995	0.995	0.995	0.995	0.995
2.60	0.995	0.995	0.996	0.996	0.996	0.996	0.996	0.996	0.996	0.996
2.70	0.997	0.997	0.997	0.997	0.997	0.997	0.997	0.997	0.997	0.997
2.80	0.997	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.998
2.90	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.999	0.999	0.999
3.00	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999

Example: If  $Z \sim \mathcal{N}(0, 1)$ , then  $\Pr(Z \leq 1.15) = F(1.15) = 0.875$ .