UCLA Fall '99 QUANTATATIVE METHODS

CORE EXAM

There are three parts to this exam. From each part choose two out of the three questions.

Choose two out of the following three questions.

1. Let the random variable X have a discrete distribution with $Pr(X = i) = 1/6$ for $i = 1, 2, \ldots, 6$. Conditional on $X = x$, the random variable Y has a normal distribution with mean $\mu + x$ and variance one.

   (a) Find the conditional probability that $X = 1$ given $Y = y$.

   (b) Calculate the mean and variance of Y (marginal, not conditional on X).

   (c) Calculate the Cramer-Rao bound for μ.

   (d) Find an unbiased estimator for μ with variance equal to the Cramer-Rao bound.

   (e) Let $(x_1, y_1), \ldots, (x_N, y_N)$ be a random sample from this distribution. Find a one-dimensional sufficient statistic for μ.

   (f) Find the maximum likelihood estimator for μ.

2. Let $X_1, X_2, \ldots, X_N$ be independent random variables with normal distributions with mean and variance both equal to θ.

   (a) Find an unbiased estimator for θ.

   (b) Is there an unbiased estimator with variance equal to the Cramer-Rao bound?

   (c) Find the maximum likelihood estimator.

   (d) Find the asymptotic variance of the maximum likelihood estimator.

   (e) Given that $N = 10$, $\sum x_i = 40$, and $\sum x_i^2 = 200$, test the null hypothesis that $\theta = 5$ at the 5% level against the alternative hypothesis that θ differs from 5 using a likelihood ratio test.

   (f) Test the same null hypothesis using a Wald test.

3. Let X be a random variable with a Gamma distribution with probability density function

$$f_X(x; \beta) = \frac{x \exp(-x/\beta)}{\beta^2},$$

   for $x > 0$ and zero elsewhere.

(a) Find the moment generating function of X.

(b) Calculate the mean and variance of X.

(c) Show that if Y and Z have independent exponential distributions with mean β, then the distribution of $W = Y + Z$ is the same as the distribution of X.

(d) Let $x_1, x_2, \ldots, x_N$ be a random sample from this distribution, with $N = 10$, $\sum_{i=1}^{N} x_i = 400$. Calculate the maximum likelihood estimate.

(e) Estimate the variance of the maximum likelihood estimator.

Choose two out of the following three questions.

1. Suppose you have n independent observations $\{(y_i, X_i)\}_{i=1}^{n}$ from the k-variate classical regression model, $y = X\beta + \varepsilon$, where q linearly independent restrictions of the form $R\beta = r$ hold.

    (a) Derive the restricted least squares estimator of β.

    (b) Derive an unbiased estimator for the variance of y_i for the restricted model.

2. For each one of the following claims show whether they are true or false.

    (a) For the simple regression model $y = \mu + \varepsilon$ with homoskedastic but equicorrelated errors (i.e. $var(\varepsilon_i) = var(\varepsilon_j)$ and $corr(\varepsilon_i, \varepsilon_j) = \rho$ for all $i \neq j$), the OLS estimator of μ is consistent.

    (b) For the k-variate regression model, $y = X\beta + \varepsilon$, the fit as measured by R^2 does not change if we transform the X matrix by postmultiplying it by a $k \times k$ non-singular matrix.

3. Consider the following time series model:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \qquad t = ..., -1, 0, 1, ...$$

   where $\phi_1 = 0.3$, $\phi_2 = -0.02$, and ε_t follows a white noise process with mean 0 and variance 1, that is uncorrelated with all past Y's.

    (a) Is the process Y_t weakly stationary?

    (b) Suppose you don't know ϕ_1 and ϕ_2. Assuming you have a sample of T observations $(Y_1, ..., Y_T)$ and you know that the ε_t's are i.i.d. with a normal distribution demonstrate how you would estimate the unknown parameters of the model. Discuss the finite sample properties (bias, variance) and asymptotic properties (consistency, asymptotic normality) of your estimator.

Choose two out of the following three questions.

1. Suppose that x_i measures years of schooling, and y_i measures earnings, for individuals $i = 1, \ldots, n$. In the sample, y_i is observed to be 0 for some individuals and positive for others. Consider the following model:

$$y_i^* = \alpha + \beta x_i + \epsilon_i,$$

$$\epsilon_i \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2),$$

$$\begin{aligned} y_i &= 0 \text{ if } y_i^* \leq 0 \\ &= y_i^* \text{ if } y_i^* > 0 \end{aligned}$$

(a) Write down the likelihood function and explain (without giving proofs) the properties of ML estimation in this case.

(b) Let $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\sigma}$ denote the ML estimates of α, β, and σ, respectively. Provide an estimator of the marginal effect of an additional year of schooling on average earnings:

$$\frac{\partial E(y_i|x_i)}{\partial x_i}.$$

Explain why your estimator is reasonable. Is the estimator consistent?

(c) Explain how to estimate the function $h(x)$ that minimizes

$$E\left[(y_i - h(x))^2\right].$$

If we are limited to using functions $h(x)$ that are linear, how would you estimate it?

2. Consider a linear model

$$y_i = x_i'\beta + \epsilon_i$$

where x_i is a $K \times 1$ vector of variables, which are possibly correlated with the disturbance ϵ_i. We have a $J \times 1$ vector ($J \geq K$) of instruments z_i, which satisfy $E(z_i\epsilon_i) = 0$ and $E(z_i x_i')$ is nonsingular.

Consider the IV estimator which solves

$$\min_{\beta} \left(\frac{1}{n}\sum z_i(y_i - x_i'\beta)\right)' A_n^{-1} \left(\frac{1}{n}\sum z_i(y_i - x_i'\beta)\right),$$

(a) If $J = K$, does the choice of A_n affect the solution of the minimization problem? Explain your reasoning.

(b) Show that 2SLS is a special case of the general estimator given above.

(c) Which choice of A_n leads to optimal estimates within the class of IV estimators, under general heteroskedasticity? If in fact the disturbances are homoskedastic, will this choice of weight matrix still be optimal?

3. Consider a firm with a Cobb-Douglas production technology

$$Y = AL^{\gamma_1}K^{\gamma_2}.$$

where $\gamma_1 + \gamma_2 < 1$.

(a) Suppose we observe Y, L, K for a random sample of firms. One might try to estimate γ_1 and γ_2 by running a least-squares regression of $\log Y$ on a constant, $\log L$, and $\log K$. Are there any conditions under which this would be a consistent and/or unbiased estimator?

(b) Briefly describe how panel data on firms can be used to estimate γ_1 and γ_2. What are the key assumptions that justify this approach?

(c) Let p_y be the price of the output, and p_l and p_k be the prices of labor and capital. Show that if the firm is a price taker and chooses inputs to maximize profits,

$$\gamma_1 = \frac{p_l L}{p_y Y}$$

and

$$\gamma_2 = \frac{p_k K}{p_y Y}$$

Suppose that you observe p_y, p_l, p_k in addition to Y, L, K for a sample of firms. Suggest an estimator for (γ_1, γ_2) that uses the price information.

Table A.2 Chi-square cumulative distribution function

	$G_k(\cdot)$										
k	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55
1	0.00	0.02	0.04	0.06	0.10	0.15	0.21	0.27	0.36	0.45	0.57
2	0.10	0.21	0.33	0.45	0.58	0.71	0.86	1.02	1.20	1.39	1.60
3	0.35	0.58	0.80	1.01	1.21	1.42	1.64	1.87	2.11	2.37	2.64
4	0.71	1.06	1.37	1.65	1.92	2.19	2.47	2.75	3.05	3.36	3.69
5	1.15	1.61	1.99	2.34	2.67	3.00	3.33	3.66	4.00	4.35	4.73
6	1.64	2.20	2.66	3.07	3.45	3.83	4.20	4.57	4.95	5.35	5.77
7	2.17	2.83	3.36	3.82	4.25	4.67	5.08	5.49	5.91	6.35	6.80
8	2.73	3.49	4.08	4.59	5.07	5.53	5.98	6.42	6.88	7.34	7.83
9	3.33	4.17	4.82	5.38	5.90	6.39	6.88	7.36	7.84	8.34	8.86
10	3.94	4.87	5.57	6.18	6.74	7.27	7.78	8.30	8.81	9.34	9.89
11	4.57	5.58	6.34	6.99	7.58	8.15	8.70	9.24	9.78	10.34	10.92
12	5.23	6.30	7.11	7.81	8.44	9.03	9.61	10.18	10.76	11.34	11.95
13	5.89	7.04	7.90	8.63	9.30	9.93	10.53	11.13	11.73	12.34	12.97
14	6.57	7.79	8.70	9.47	10.17	10.82	11.45	12.08	12.70	13.34	14.00
15	7.26	8.55	9.50	10.31	11.04	11.72	12.38	13.03	13.68	14.34	15.02
16	7.96	9.31	10.31	11.15	11.91	12.62	13.31	13.98	14.66	15.34	16.04
17	8.67	10.09	11.12	12.00	12.79	13.53	14.24	14.94	15.63	16.34	17.06
18	9.39	10.86	11.95	12.86	13.68	14.44	15.17	15.89	16.61	17.34	18.09
19	10.12	11.65	12.77	13.72	14.56	15.35	16.11	16.85	17.59	18.34	19.11
20	10.85	12.44	13.60	14.58	15.45	16.27	17.05	17.81	18.57	19.34	20.13

Example: If $W \sim \chi^2(6)$, then $\Pr(W \leq 4.20) = G_6(4.20) = 0.35$.

Table A.2 (continued)

	$G_k(\cdot)$										
k	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	0.975	0.990	0.9
1	0.71	0.87	1.07	1.32	1.64	2.07	2.71	3.84	5.02	6.63	7.
2	1.83	2.10	2.41	2.77	3.22	3.79	4.61	5.99	7.38	9.21	10.
3	2.95	3.28	3.66	4.11	4.64	5.32	6.25	7.81	9.35	11.34	12.
4	4.04	4.44	4.88	5.39	5.99	6.74	7.78	9.49	11.14	13.28	14.
5	5.13	5.57	6.06	6.63	7.29	8.12	9.24	11.07	12.83	15.09	16.
6	6.21	6.69	7.23	7.84	8.56	9.45	10.64	12.59	14.45	16.81	18.
7	7.28	7.81	8.38	9.04	9.80	10.75	12.02	14.07	16.01	18.48	20.
8	8.35	8.91	9.52	10.22	11.03	12.03	13.36	15.51	17.53	20.09	21.
9	9.41	10.01	10.66	11.39	12.24	13.29	14.68	16.92	19.02	21.67	23.
10	10.47	11.10	11.78	12.55	13.44	14.53	15.99	18.31	20.48	23.21	25.1
11	11.53	12.18	12.90	13.70	14.63	15.77	17.28	19.68	21.92	24.72	26.7
12	12.58	13.27	14.01	14.85	15.81	16.99	18.55	21.03	23.34	26.22	28.[illegible]
13	13.64	14.35	15.12	15.98	16.98	18.20	19.81	22.36	24.74	27.69	29.8
14	14.69	15.42	16.22	17.12	18.15	19.41	21.06	23.68	26.12	29.14	31.3
15	15.73	16.49	17.32	18.25	19.31	20.60	22.31	25.00	27.49	30.58	32.8
16	16.78	17.56	18.42	19.37	20.47	21.79	23.54	26.30	28.85	32.00	34.2
17	17.82	18.63	19.51	20.49	21.61	22.98	24.77	27.59	30.19	33.41	35.7
18	18.87	19.70	20.60	21.60	22.76	24.16	25.99	28.87	31.53	34.81	37.1
19	19.91	20.76	21.69	22.72	23.90	25.33	27.20	30.14	32.85	36.19	38.5
20	20.95	21.83	22.77	23.83	25.04	26.50	28.41	31.41	34.17	37.57	40.0