Department of Economics **Fall 2015**
UCLA

Comprehensive Examination
Quantitative Methods

This exam consists of three parts. You are required to answer all the questions in all the parts. Each part is worth 100 points, with relative weights given by the points by each question. Allocate your time wisely. Good luck!

Part I - 203A

Question 1 (20 points)

Suppose that A1, A2, B1, and B2 are four mutually independent random variables, distributed with, respectively, marginal cumulative distribution functions, F_{A_1}, F_{A_2}, F_{B_1}, and F_{B_2}. Assume that F_{A_1}, F_{A_2}, F_{B_1}, and F_{B_2} are strictly increasing and differentiable. Let

$$A = \min\{A_1, A_2\} \quad \text{and} \quad B = \min\{B_1, B_2\}$$

and

$$C = \max\{A, B\}$$

Answer the following questions in terms of F_{A_1}, F_{A_2}, F_{B_1}, and F_{B_2}.

(a; 10 points) Derive the probability densities of (A, B) and of C. Are A and B independent? Justify your answers.

(b; 10 points) Let $E = (A_1)^2 + (B_1)^2$. Derive the density of E.

Question 2 (80 points):

Consider the following model

$$\begin{aligned} Y_1 &= \alpha + \beta\ Y_2 + \varepsilon_1 \\ Y_2 &= \delta + \gamma\ X + \varepsilon_2 \end{aligned}$$

where α, β, δ, and γ are parameters, the probability density of ε_2 conditional on $X = x$ is, for a parameter $\sigma_2 > 0$,

$$f_{\varepsilon_2|X=x}(\varepsilon_2) = \frac{1}{\sqrt{2\pi\sigma_2^2}}\ e^{-\frac{1}{2}\left(\frac{\varepsilon_2 - x}{\sigma_2}\right)^2} \qquad -\infty < \varepsilon_2 < \infty$$

and the probability density of ε_1 is , for a parameter $\sigma_1 > 0$

$$f_{\varepsilon_1}(\varepsilon_1) = \frac{1}{\sqrt{2\pi\sigma_1^2}}\ e^{-\frac{1}{2}\left(\frac{\varepsilon_1}{\sigma_1}\right)^2} \qquad -\infty < \varepsilon_1 < \infty$$

The random variable ε_1 is assumed to be distributed independently of X, and the (marginal) density of X is given by

$$f_X(x) = \left\{ \begin{array}{ll} \frac{1}{(\theta_2 - \theta_1)} & \theta_1 < x < \theta_2 \\ & \\ 0 & \text{otherwise} \end{array} \right\}$$

for parameters θ_1 and θ_2.

The answers to (a)-(c) should be in terms of parameters.

(a; 10 points) What is the density of Y_1 conditional on $X = x$ for x in (θ_1, θ_2)? What is the expectation of Y_1?

(b; 10 points) What is the probability that $(Y_2 \geq 0)$? What is the probability that $(X \geq 0)$ conditional on ε_2?

(c; 10 points) What is the joint cumulative distribution of (Y_1, Y_2)? What is the moment generating function of Y_2?

Suppose now that the density of ε_2 conditional on $X = x$, for x in (θ_1, θ_2), is an unknown continuous function, $f_{\varepsilon_2|X=x}$, of ε_2 and x, and as above,

$$Y_2 = \delta + \gamma\ X + \varepsilon_2$$

(d; 15 points) What can and what cannot be identified about the function $f_{\varepsilon_2|X=x}$ from the distribution of (Y_2, X) under the following cases?

(i) No additional information.

(ii) $\delta = 0$ and $\gamma = 1$.

(iii) For all for x in (θ_1, θ_2), the expectation of ε_2 given $X = x$ is 0.

(iv) For all for x in (θ_1, θ_2).

$$\int_{-\infty}^{0} f_{\varepsilon_2|X=x}(t)\ dt = 0.5$$

Justify your answers.

(e; 15 points) What can and what cannot be identified about the function $f_{\varepsilon_2|X=x}$ from the probability that $(Y_2 \geq 0)$ conditional on $X = x$ for all x in (θ_1, θ_2) under the following cases?

(i) No additional information.
(ii) $\delta = 0$ and $\gamma = 1$.
(iii) For all for x in (θ_1, θ_2), the expectation of ε_2 given $X = x$ is 0.
(iv) ε and X are distributed independently.

Justify your answers.

Suppose that you can observe an i.i.d. sample $\{X_1, X_2, ...\}$ of size N from the distribution of X.

(f; 20 points) Assume that $\theta_1 > 0$. Let $\phi = \sqrt{(\theta_1 + \theta_2)}$. Provide a consistent estimator for ϕ and an approximate confidence interval, in terms of the observations.

Part II - 203B

Question 1

Suppose that

$$\begin{aligned} y_i &= x_i\beta + \varepsilon_i \\ x_i &= \frac{1}{2}z_{i1} + \frac{1}{3}z_{i2} + v_i, \end{aligned}$$

where we assume that (i) $(z_{i1}, z_{i2})'$ is independent of $(\varepsilon_i, v_i)'$; (ii) z_{i1} and z_{i2} are independent of each other; (iii) $E[z_{i1}] = E[z_{i2}] = E[\varepsilon_i] = E[v_i] = 0$; (iv) $E[z_{i1}^2] = E[z_{i2}^2] = E[\varepsilon_i^2] = E[v_i^2] = 1$; and (v) $(x_i, z_{i1}, z_{i2}, \varepsilon_i, v_i)'$ $i = 1, 2, \ldots$ are i.i.d.

Question 1-1 (10 pts.)

No derivation is required for this question; your derivation will not be read anyway.

Let

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^n z_{i1}y_i}{\sum_{i=1}^n z_{i1}x_i}, \quad \widehat{\beta}_2 = \frac{\sum_{i=1}^n z_{i2}y_i}{\sum_{i=1}^n z_{i2}x_i}$$

It can be shown that

$$\begin{bmatrix} \sqrt{n}\left(\widehat{\beta}_1 - \beta\right) \\ \sqrt{n}\left(\widehat{\beta}_2 - \beta\right) \end{bmatrix}$$

is asymptotically normal with mean zero. Provide numerical characterization of the asymptotic variance matrix. (You do not have to establish asymptotic normality. Just state the asymptotic variance matrix. Your answer should take the form of numbers; an abstract formula will not be accepted as an answer.)

Question 1-2 (10 pts.)

No derivation is required for this question; your derivation will not be read anyway.

Let

$$\begin{aligned} \widehat{\beta} &= \frac{\left(\sum_i x_i z_i'\right)\left(\sum_i z_i z_i'\right)^{-1}\left(\sum_i z_i y_i\right)}{\left(\sum_i x_i z_i'\right)\left(\sum_i z_i z_i'\right)^{-1}\left(\sum_i z_i x_i\right)} \\ &= \beta + \frac{\left(\frac{1}{n}\sum_i x_i z_i'\right)\left(\frac{1}{n}\sum_i z_i z_i'\right)^{-1}\left(\frac{1}{n}\sum_i z_i \varepsilon_i\right)}{\left(\frac{1}{n}\sum_i x_i z_i'\right)\left(\frac{1}{n}\sum_i z_i z_i'\right)^{-1}\left(\frac{1}{n}\sum_i z_i x_i\right)} \end{aligned}$$

denote 2SLS. It can be shown that $\sqrt{n}\left(\widehat{\beta} - \beta\right)$ is asymptotically normal with mean zero. Provide numerical characterization of the asymptotic variance. (You do not have to establish asymptotic normality. Just state the asymptotic variance. Your answer should take the form of a number; an abstract formula will not be accepted as an answer.)

Question 2 (10 pts.)

No derivation is required for this question; your derivation will not be read anyway.

Consider the following two-equation model:

$$\begin{aligned} y_1 &= \gamma_1 y_2 + \beta_{11} x_1 + \beta_{31} x_3 + \varepsilon_1 \\ y_2 &= \gamma_2 y_1 + \beta_{12} x_1 + \beta_{22} x_2 + \varepsilon_2 \end{aligned}$$

where we assume that

$$E\begin{bmatrix} x_1\varepsilon_1 \\ x_2\varepsilon_1 \\ x_3\varepsilon_1 \end{bmatrix} = E\begin{bmatrix} x_1\varepsilon_2 \\ x_2\varepsilon_2 \\ x_3\varepsilon_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We know that

$$\begin{aligned} y_1 &= x_1 + 2x_2 + 3x_3 + u_1 \\ y_2 &= 4x_1 + 5x_2 + 6x_3 + u_2 \end{aligned}$$

where

$$E\begin{bmatrix} x_1 u_1 \\ x_2 u_1 \\ x_3 u_1 \end{bmatrix} = E\begin{bmatrix} x_1 u_2 \\ x_2 u_2 \\ x_3 u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

What are the numerical values of γs and βs?

Question 3 (20 pts.)

No derivation is required for this question; your derivation will not be read anyway.

Suppose that

$$y_i = \alpha + \beta x_i + \varepsilon_i$$

with $(y_i, x_i)'$ $i = 1, \ldots, n$ i.i.d., and $E[\varepsilon_i] = E[x_i \varepsilon_i] = 0$. You are given the following data set with $n = 3$:

$$\begin{bmatrix} y_1 & x_1 \\ y_2 & x_2 \\ y_3 & x_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 4 & 2 \end{bmatrix}$$

The OLS estimates from a regression of y on a constant and x using the given data is

$$\begin{bmatrix} \widehat{\alpha} \\ \widehat{\beta} \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the residual vector is

$$e = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Assuming heteroskedasticity and using White's heteroscedasticity corrected asymptotic variance estimator, provide 95% asymptotic confidence interval for β. If your answer involves a square root, try to simplify as much as you can.

Question 4 (10 pts.)

No derivation is required for this question; your derivation will not be read anyway.

Suppose that

$$y_i = \beta_i x_i + \varepsilon_i \qquad i = 1, \ldots, n$$

and that (i) $(\beta_i, x_i, \varepsilon_i)$ $i = 1, 2, \ldots$ are i.i.d.; (ii) x_i, β_i, and ε_i are independent of each other; (iii) $x_i \sim N(0,1)$, $\beta_i \sim N(\beta, 1)$, $\varepsilon_i \sim N(0,1)$. Note that $\beta = E[\beta_i]$. Let

$$\widehat{\beta} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$

denote the coefficient of x_i when y_i is regressed on x_i. What is the asymptotic distribution of $\sqrt{n}\left(\widehat{\beta} - \beta\right)$? Make sure that the asymptotic variance is a concrete number, not an abstract formula.

Question 5 (20 pts.)

In this question, your derivation as well as your answer will be read and evaluated.

Suppose that we have

$$\begin{aligned} y_i &= \alpha + \beta x_i + \varepsilon_i \qquad & i = 1, \ldots, 10 \\ y_i &= \gamma + \delta x_i + \varepsilon_i \qquad & i = 11, \ldots, 20 \end{aligned}$$

We assume that $x_1, \ldots, x_{20}$ are nonstochastic, and that ε_i are i.i.d. such that $\varepsilon_i \sim N(0,1)$. We assume that

$$\sum_{i=1}^{10} x_i = 0, \qquad \sum_{i=11}^{20} x_i = 0$$

$$\sum_{i=1}^{10} x_i^2 = 20, \qquad \sum_{i=11}^{20} x_i^2 = 40$$

Let $\left(\widehat{\alpha}, \widehat{\beta}\right)$ and $\left(\widehat{\gamma}, \widehat{\delta}\right)$ denote the OLS coefficients when y is regressed on a constant and x for $i = 1, \ldots, 10$ and $i = 11, \ldots, 20$. What is the distribution of $\left(\widehat{\delta} - \widehat{\beta}\right) - (\delta - \beta)$? (Your will get at most 50% of the credit if you do not establish the joint distribution of $\left(\widehat{\delta}, \widehat{\beta}\right)$ rigorously.)

Question 6 (20 pts.)

In this question, your derivation as well as your answer will be read and evaluated.

Suppose that

$$y_i = \beta_i x_i + \varepsilon_i \qquad i = 1, \ldots, n$$

and that (i) $(\beta_i, x_i, \varepsilon_i)$ $i = 1, 2, \ldots$ are i.i.d.; (ii) x_i is independent of ε_i; (iii) $x_i \sim N(1,1)$; and (iv) $\beta_i | x_i \sim N(x_i, 1)$ and $\varepsilon_i \sim N(0,1)$ are independent of each other. Let $\widehat{\beta}$ denote the coefficient of x_i when y_i is regressed on x_i. What is the numerical value of $\text{plim}\left(\widehat{\beta} - E[\beta_i]\right)$? You may want to recall that the moment generating function of $N(\mu, \sigma^2)$ is $\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$.

Part III - 203C

1. A random variable X is called exponential(λ) if it has the probability density function

$$f(x) = \begin{cases} \lambda \exp(-\lambda x), & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Suppose that we have two independent random samples: $\{X_i\}_{i=1}^n$ from exponential(θ) and $\{Y_j\}_{j=1}^m$ from exponential(μ).

(a) (10 points) Find the likelihood ratio test of H_0: $\theta = \mu$ v.s. H_1: $\theta \neq \mu$.

(b) (10 points) Show that the test in part (a) can be based on the statistic

$$T_{n,m} = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{j=1}^m Y_j}.$$

(c) (5 points) Find the finite sample distribution of $T_{n,m}$ when H_0 is true. Moreover, find the critical value of the level-α test in part (a) based on $T_{n,m}$.

(d) (10 points) Find the asymptotic properties of the level-α test in (c) under H_0 in the following three separate scenarios: (i) $n/m \to 0$; (ii) $n/m \to \infty$; and (iii) $n/m \to c \in (0, \infty)$.

(e) (10 points) Find the asymptotic properties of the level-α test in (c) under H_1 in the following three separate scenarios: (i) $n/m \to 0$; (ii) $n/m \to \infty$; and (iii) $n/m \to c \in (0, \infty)$.

2. Suppose $\{X_t\}$ is generated from the following model

$$X_t = \theta X_{t-1} + u_t + \gamma u_{t-1},$$

where u_t is *i.i.d.* $(0, \sigma_u^2)$ with finite 4-th moment and $|\theta| < 1$. We have T observations $\{X_t\}_{t=1}^T$ from this process.

(a) (10 points) Find the spectral density of the process $\{X_t\}$.

(b) (10 points) Show that $E\left[(X_t - \theta X_{t-1})X_{t-2}\right] = 0$ for any $|\theta| < 1$ and for any γ. Construct the method of moment (MM) estimator of θ and find its asymptotic distribution.

(c) (10 points) Suppose that one is interested in testing H_0: $\theta = 0$ versus H_1: $\theta \neq 0$. Construct a test using the MM estimator in part (b) and study its asymptotic properties under both H_0 and H_1.

(d) (10 points) Suppose that we know that $\theta \in (0,1)$ and $\gamma > 0$. Prove or disprove the following statement: θ and γ are identified by the following moment conditions

$$\begin{array}{c} E\left[(X_t - \theta X_{t-1})X_{t-2}\right] = 0 \\ E\left[(X_t - (\theta+\gamma)X_{t-1} + \gamma\theta X_{t-2})X_{t-3}\right] = 0 \end{array} .$$

(e) (15 points) Consider the LS estimator of θ:

$$\widehat{\theta}_n = \frac{\sum_{t=2}^{T} X_t X_{t-1}}{\sum_{t=1}^{T} X_t^2}.$$

From $\widehat{\theta}_n$, we can construct the fitted residual $\widehat{u}_t = X_t - \widehat{\theta}_n X_{t-1}$ for $t = 2, \ldots, T$, and then construct the following statistic

$$\widehat{\rho}_n = \frac{\sum_{t=3}^{T} \widehat{u}_t \widehat{u}_{t-1}}{\sum_{t=2}^{T} \widehat{u}_t^2}.$$

Under the conditions in (d), find the probability limit of $\widehat{\rho}_n$.

Some Useful Theorems and Lemmas

Theorem 1 *If $X_1, X_2, \ldots, X_k$ are i.i.d. from exponential(λ), then $\sum X_i$ is a gamma(k, λ) random variable. Moreover, if X is a gamma(k_1, λ) random variable, Y is a gamma(k_2, λ) random variable, and X and Y are independent, then $X/(X+Y)$ is a beta(k_1, k_2) random variable whose pdf only depends on k_1 and k_2.*

Theorem 2 (Martingale Convergence Theorem) *Let $\{(X_t, \mathcal{F}_t)\}_{t \in \mathbb{Z}_+}$ be a martingale in L^2. If $\sup_t E\left[|X_t|^2\right] < \infty$, then $X_n \to X_\infty$ almost surely, where X_∞ is some element in L^2.*

Theorem 3 (Martingale CLT) *Let $\{X_{t,n}, \mathcal{F}_{t,n}\}$ be a martingale difference array such that $E[|X_{t,n}|^{2+\delta}] < \Delta < \infty$ for some $\delta > 0$ and for all t and n. If $\overline{\sigma}_n^2 > \delta_1 > 0$ for all n sufficiently large and $\frac{1}{n}\sum_{t=1}^n X_{t,n}^2 - \overline{\sigma}_n^2 \to_p 0$, then $n^{\frac{1}{2}}\overline{X}_n/\overline{\sigma}_n \to_d N(0,1)$.*

Theorem 4 (LLN of Linear Processes) *Suppose that Z_t is i.i.d. with mean zero and $E[|Z_0|] < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k\,|\varphi_k| < \infty$. Then $n^{-1}\sum_{t=1}^n X_t \to_{a.s.} 0$.*

Theorem 5 (CLT of Linear Processes) *Suppose that Z_t is i.i.d. with mean zero and $E[Z_0^2] = \sigma_Z^2 < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k^2\varphi_k^2 < \infty$. Then $n^{-\frac{1}{2}}\sum_{t=1}^n X_t \to_d N\left[0, \varphi(1)^2\sigma_Z^2\right]$.*

Theorem 6 (LLN of Sample Variance) *Suppose that Z_t is i.i.d. with mean zero and $E[Z_0^2] = \sigma_Z^2 < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k\varphi_k^2 < \infty$. Then*

$$\frac{1}{n}\sum_{t=1}^n X_t X_{t-h} \to_p \Gamma_X(h) = E\left[X_t X_{t-h}\right]. \tag{1}$$

Theorem 7 (Donsker) *Let $\{u_t\}$ be a sequence of random variables generated by $u_t = \sum_{k=0}^\infty \varphi_k \varepsilon_{t-k} = \varphi(L)\varepsilon_t$, where $\{\varepsilon_t\} \sim iid\ (0, \sigma_\varepsilon^2)$ with finite fourth moment and $\{\varphi_k\}$ is a sequence of constants with $\sum_{k=0}^\infty k\,|\varphi_k| < \infty$. Then $B_{u,n}(\cdot) = n^{-\frac{1}{2}}\sum_{t=1}^{[n\cdot]} u_t \to_d \lambda B(\cdot)$, where $\lambda = \sigma_\varepsilon\varphi(1)$.*