Department of Economics
UCLA

Fall 2014

Comprehensive Examination
Quantitative Methods

This exam consists of three parts. You are required to answer all the questions in all the parts. For each part, the relative weight of each question can be inferred from the relation of the associated "points" to the total number of points (60). Allocate your time wisely. Good luck!

Part I - 203A

An individual must choose one out of a set of J alternatives, {1, ..., J}. For each individual i and alternative j, individual i chooses alternative j if

$$\eta^i_{jk} = V^i_j - V^i_k > 0 \qquad \text{for } k \neq j$$

a. (10 points) Suppose that $J = 3$ and for each i, the random vector (V^i_1, V^i_2, V^i_3) has a Normal distribution with mean (μ_1, μ_2, μ_3) and variance

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

a.1. Provide an expression, in terms of the parameters, for the probability that individual i chooses alternative 1.

a.2. Provide an expression, in terms of the parameters, for the probability that individual i chooses alternative 1, given that it is known that his/her choice is either alternative 1 or alternative 2.

b. (10 points) Suppose that for each j

$$V^i_j = \alpha_j + \beta_j X^i_j$$

where α_j and β_j are parameters, and (X^i_1, X^i_2, X^i_3) is distributed Normal with mean $(0, 0, 0)$ and variance

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

b.1. Provide an expression, in terms of the parameters, for the probability that $(V_2^i > V_3^i)$, when it is known that the choice of individual i was alternative 1.

b.2. Provide an expression, in terms of the parameters, for the probability that $(X_1^i < 0)$, when it is known that the choice of individual individual i was alternative 1.

c. (25 points) Suppose that the number of alternatives, J, is 2, and that the utility of individual i for alternative j (j=1,2) is

$$V_j^i = m_j(X_j^i) + \varepsilon_j^i$$

where $(\varepsilon_1^i, \varepsilon_2^i)$ is distributed independently of (X_1^i, X_2^i) with the following density

$$f_{\varepsilon_1,\varepsilon_2}(\varepsilon_1, \varepsilon_2) = \begin{cases} c^2 \; e^{-c\;\varepsilon_2} & if \; 0 < \varepsilon_1 < \varepsilon_2 \\ \\ 0 & otherwise \end{cases}$$

for $c > 0$, and where the density of (X_1^i, X_2^i) is

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 1 & if \quad 0 < x_1 < 1 \quad and \quad 0 < x_2 < 1 \\ \\ 0 & otherwise \end{cases}$$

c.1. Obtain an expression for the joint density of $(V_1^i, V_2^i, X_1^i, X_2^i)$.

c.2. Obtain an expression for the density of $\eta = \varepsilon_2^i - \varepsilon_1^i$.

c.3. Suppose that the functions m_1, m_2 and the parameter c are unknown. Suppose also that the probability that individual i chooses alternative 1 given the observation that $(X_1^i, X_2^i) = (x_1, x_2)$ is known for all (x_1, x_2) in the support of (X_1, X_2). Is c identified? If your answer Yes, provide a proof. If your answer is No, justify, provide conditions under which identification is achieved, and provide a proof.

c.4. Suppose that the functions m_1, m_2 are unknown and the parameter c is known. Suppose also that the probability that individual i chooses alternative 1 given the observation that $(X_1^i, X_2^i) = (x_1, x_2)$ is known for all (x_1, x_2) in the support of (X_1, X_2). Are the functions m_1 and m_2 identified? If your answer Yes, provide a proof. If your answer is No, justify, provide conditions under which identification is achieved, and provide a proof.

d. (15 points) Suppose that $J = 2$, the cummulative distribution of $(\varepsilon_2 - \varepsilon_1)$ is a known, strictly increasing and continuously differentiable function F, and for each i, alternative 1 is chosen by i if $V_1^i > V_2^i$, where $V_1^i = \mu + \varepsilon_1^i$ and $V_2^i = \varepsilon_2^i$. The value of μ is unknown. An iid sample $\{i = 1, ..., N\}$ is drawn, and you are told that the value of $\overline{Y}_N = (1/N) \sum_{i=1}^{N} Y^i$, where for each i, $Y^i = 1$ if individual i chose alternative 1 and $Y^i = 0$ otherwise. Suggest a consistent estimator for μ, using $\overline{Y}_N$. Prove that the estimator is consistent and derive its asymptotic distribution.

Part II - 203B

Question II-1: Short Questions

No derivation is required for the short questions below; your derivation will not be read anyway.

Question II-1 (a, 10 points)

Suppose that

$$y_i = x_i\beta + z_i\gamma + \varepsilon_i$$

with $(y_i, x_i, z_i)'$ $i = 1, \ldots, n$ i.i.d., and $E[x_i\varepsilon_i] = E[z_i\varepsilon_i] = 0$. You do not observe z_i, so you regressed y_i on x_i. Let $\widehat{\beta}$ denote such OLS estimator. Derive plim $\left(\widehat{\beta} - \beta\right)$ under the assumption that $\gamma = 5$ and

$$\begin{bmatrix} x_i \\ z_i \end{bmatrix} \sim N\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}\right).$$

Question II-1 (b, 10 points)

Suppose that

$$y_i = x_i\beta + \varepsilon_i$$

with $(y_i, x_i, z_i)'$ $i = 1, \ldots, n$ i.i.d., and z_i are independent of ε_i, and $E[\varepsilon_i] = 0$. You are given the following data set with $n = 3$:

$$\begin{bmatrix} y_1 & x_1 & z_1 \\ y_2 & x_2 & z_2 \\ y_3 & x_3 & z_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 1 & 3 \end{bmatrix}$$

Using the asymptotic normality of the IV estimator, produce a 95% confidence interval of β. If you need to deal with a square root, try to simplify as much as you can.

Question II-1 (c, 10 points)

Suppose that

$$y_i = x_i\beta + \varepsilon_i$$

We observe $(y_i, x_i, z_i)'$, $i = 1, \ldots, n$ i.i.d. We assume that (1) the random vector $(x_i, z_i)'$ is independent of ε_i; (2) $E[\varepsilon_i] = 0$ and $E[\varepsilon_i^2] = 1 < \infty$; (3) $E[x_i^2] = 1$, $E[z_i^2] = 1$ and $E[x_i z_i] = \frac{1}{2}$. Provide the asymptotic distribution of

$$\sqrt{n}\left(\widehat{\beta}_{IV} - \widehat{\beta}_{OLS}\right),$$

where

$$\widehat{\beta}_{OLS} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}, \quad \widehat{\beta}_{IV} = \frac{\sum_{i=1}^{n} z_i y_i}{\sum_{i=1}^{n} z_i x_i}.$$

Hint:

$$\left[\begin{array}{c} \sqrt{n}\left(\widehat{\beta}_{OLS} - \beta\right) \\ \sqrt{n}\left(\widehat{\beta}_{IV} - \widehat{\beta}_{OLS}\right) \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right] \left[\begin{array}{c} \sqrt{n}\left(\widehat{\beta}_{OLS} - \beta\right) \\ \sqrt{n}\left(\widehat{\beta}_{IV} - \beta\right) \end{array} \right]$$

Question II-1 (d, 10 points)

Suppose

$$\begin{aligned} y_i^* &= x_i'\beta + \varepsilon_i \\ \varepsilon_i | \, x_i &\sim N\left(0, 2^2\right) \end{aligned}$$

We observe (y_i, D_i, x_i),where

$$\begin{aligned} D_i &= 1\left(y_i > 0\right) \\ y_i &= D_i \cdot y_i^* \end{aligned}$$

The x_i consists of two components, i.e., it is a two dimensional vector. What is

$$E\left[y_i | \, x_i' = (0,0), D_i = 1 \right]?$$

In case you have forgotten, the PDF $\phi(t)$ of $N(0,1)$ is given by

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right)$$

Question II-2 (20 points)

Your answer to this question will be evaluated based on the logical validity of your argument as well as the accuracy of your answer. If you stop with abstract formulae or your answers are incorrect, your answer will be understood to be at best 30% complete.

Suppose that

$$y_i = x_i\beta + \varepsilon_i$$

We assume that

1. x_i is independent of ε_i
2. $\varepsilon_i \sim N(0,1)$

3. $x_i \sim N(\beta, 1)$

4. $(x_i, \varepsilon_i)'$ is iid.

What is the Fisher information contained in one observation (y_i, x_i)? (10 points) Propose an estimator whose asymptotic variance is equal to the inverse of the Fisher information. (10 points) Your estimator should take the form of an explicit analytic function of the data. *If you simply write that MLE (or some solution to a maximization problem) is your proposed estimator, you will get no credit at all for this portion of the question.*

Part III - 203C

1. (20 points) In the first order autoregression:

$$Y_\theta = \alpha Y_{\theta-1} + u_\theta, \ \ |\alpha| < 1$$

the u_θ's are identically and independently distributed as $N(0, \sigma^2)$. The index θ records time which is measured in a *unit of 6 months* and Y_θ represents a time series variable. It is assumed that T *annual observations* are available on the time series variable Y.

(a) (5 points) Show that the successive *annual* observations Y_t are related by the equation:

$$Y_t = \alpha^2 Y_{t-1} + v_t \tag{1}$$

where v_t is a moving average error.

(b) (5 points) Let $\widehat{\alpha}^2$ be the estimator of α^2 obtained by applying ordinary least squares to (1) and using the observations $\{Y_t\}_{t=1}^T$. Derive the probability limit of $\widehat{\alpha}^2$ as $T \to \infty$ and explain why $\widehat{\alpha}^2$ is inconsistent.

(c) (5 points) Suggest an alternative procedure which will provide a consistent estimator of α^2 using the T annual observations $\{Y_t\}_{t=1}^T$. Justify the use of your procedure.

(d) (5 points) Show that the moving average error v_t in (1) can be represented as

$$v_t = \varepsilon_t + \lambda \varepsilon_{t-1} \tag{2}$$

where the ε_t are serially independent random variables with zero mean and variance given by σ^2/λ and

$$\lambda = 3 - \sqrt{8}.$$

2. (40 points) Consider the following location model:

$$Y_t = \theta + u_t \text{ with } u_t = \rho u_{t-1} + \varepsilon_t \tag{3}$$

where $|\rho| < 1$, ε_t's are identically and independently distributed as $N(0, \sigma^2)$ and $\sigma^2 \in (0, \infty)$. The parameters θ, ρ and σ^2 are unknown. We have T observations on Y_t: $\{Y_t\}_{t=1}^T$.

(a) (5 points) Show that the unknown parameter θ can be consistently estimated by

$$\widehat{\theta}_T = \frac{1}{T}\sum_{t=1}^{T} Y_t. \tag{4}$$

Derive the asymptotic distribution of $\widehat{\theta}_T$.

(b) (5 points) Using the estimator $\widehat{\theta}_T$ defined in (4), we can construct the fitted residual

$$\widehat{u}_t = Y_t - \widehat{\theta}_T.$$

Show that the following estimator:

$$\widehat{\rho}_T = \frac{\sum_{t=2}^{T} \widehat{u}_t \widehat{u}_{t-1}}{\sum_{t=1}^{T} \widehat{u}_t^2} \tag{5}$$

is a consistent estimator of ρ. Derive the convergence rate of $\widehat{\rho}_T$.

(c) (10 points) Derive the probability limit for the following estimator:

$$\widehat{\sigma}_{u,T}^2 = \frac{1}{T} \sum_{t=1}^{T} \widehat{u}_t^2.$$

Construct a root-T consistent estimator of σ^2.

(d) (5 points) We are interested in testing:

$$H_0 : \theta = 0 \text{ v.s. } H_1 : \theta \neq 0.$$

Construct a test which has asymptotic size 0.05. Analyze the asymptotic power of your test.

(e) (10 points) We are interested in testing:

$$H_0 : \rho = 0 \text{ v.s. } H_1 : \rho \neq 0.$$

Construct a test which has asymptotic size 0.05. Analyze the asymptotic power of your test.

(f) (5 points) Suppose that $\rho = 1$. Study the asymptotic properties of the estimator $\widehat{\theta}_T$ of θ defined in (4).

Some Useful Theorems and Lemmas

Theorem 1 (Martingale Convergence Theorem) *Let* $\{(X_t, \mathcal{F}_t)\}_{t\in\mathbb{Z}_+}$ *be a martingale in* L^2*. If* $\sup_t E\left[|X_t|^2\right] < \infty$*, then* $X_n \to X_\infty$ *almost surely, where* X_∞ *is some element in* L^2*.*

Theorem 2 (Martingale CLT) *Let* $\{X_{t,n}, \mathcal{F}_{t,n}\}$ *be a martingale difference array such that* $E[|X_{t,n}|^{2+\delta}] < \Delta < \infty$ *for some* $\delta > 0$ *and for all* t *and* n*. If* $\overline{\sigma}_n^2 > \delta_1 > 0$ *for all* n *sufficiently large and* $\frac{1}{n}\sum_{t=1}^n X_{t,n}^2 - \overline{\sigma}_n^2 \to_p 0$*, then* $n^{\frac{1}{2}}\overline{X}_n/\overline{\sigma}_n \to_d N(0,1)$*.*

Theorem 3 (LLN of Linear Processes) *Suppose that* Z_t *is i.i.d. with mean zero and* $E[|Z_0|] < \infty$*. Let* $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$*, where* φ_k *is a sequence of real numbers with* $\sum_{k=0}^\infty k\,|\varphi_k| < \infty$*. Then* $n^{-1}\sum_{t=1}^n X_t \to_{a.s.} 0$*.*

Theorem 4 (CLT of Linear Processes) *Suppose that* Z_t *is i.i.d. with mean zero and* $E[Z_0^2] = \sigma_Z^2 < \infty$*. Let* $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$*, where* φ_k *is a sequence of real numbers with* $\sum_{k=0}^\infty k^2\varphi_k^2 < \infty$*. Then* $n^{-\frac{1}{2}}\sum_{t=1}^n X_t \to_d N\left[0, \varphi(1)^2\sigma_Z^2\right]$*.*

Theorem 5 (LLN of Sample Variance) *Suppose that* Z_t *is i.i.d. with mean zero and* $E[Z_0^2] = \sigma_Z^2 < \infty$*. Let* $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$*, where* φ_k *is a sequence of real numbers with* $\sum_{k=0}^\infty k\varphi_k^2 < \infty$*. Then*

$$\frac{1}{n}\sum_{t=1}^n X_t X_{t-h} \to_p \Gamma_X(h) = E\left[X_t X_{t-h}\right]. \tag{6}$$

Theorem 6 (Donsker) *Let* $\{u_t\}$ *be a sequence of random variables generated by* $u_t = \sum_{k=0}^\infty \varphi_k\varepsilon_{t-k} = \varphi(L)\varepsilon_t$*, where* $\{\varepsilon_t\} \sim iid\ (0, \sigma_\varepsilon^2)$ *with finite fourth moment and* $\{\varphi_k\}$ *is a sequence of constants with* $\sum_{k=0}^\infty k\,|\varphi_k| < \infty$*. Then* $B_{u,n}(\cdot) = n^{-\frac{1}{2}}\sum_{t=1}^{[n\cdot]} u_t \to_d \lambda B(\cdot)$*, where* $\lambda = \sigma_\varepsilon\varphi(1)$*.*