First Year Quantitative Comp Exam
Fall, 2013

Instruction: There are three parts. Answer every question in every part. A necessary condition for passing at the PhD Level is that you pass at least two parts at the PhD Level.

Part I - 203A

Question I-1

Consider the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ that are degenerate, alternating between 0 and $1/n$ according to the following definition

$$X_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

1. Show that $X_n \xrightarrow{d} 0$ (i.e. it converges to the degenerate random variable equal to 0)
2. Compute the cumulative distribution function for X_n evaluated at 0. Does this converge as $n \to \infty$? Can you reconcile your answer to this part with your answer in the previous part?

Question I-2

Let T_n be an estimator for a parameter $\theta \in \{\theta_1, \ldots, \theta_k\}$ where k is a fixed known positive integer and $T_n \in \{\theta_1, \ldots, \theta_k\}$ (so that T_n only takes on a finite number of values).

1. Show that $T_n \xrightarrow{p} \theta \iff \mathbb{P}(T_n = \theta) \to 1$
2. Show that $n(T_n - \theta) \xrightarrow{p} 0$

Question I-3

Let $(X_1, \ldots, X_n)$ be an i.i.d. sample from a $\mathcal{N}(\mu, 1)$ distribution where $\mu \in \mathbb{R}$ is unknown. Let the parameter of interest be

$$\theta \equiv \mathbb{P}(X_1 < c) = \Phi(c - \mu)$$

for some known constant c and where Φ is the CDF of the standard normal distribution.

1. Find the MLE of θ and derive its asymptotic distribution.
2. Consider the estimator
$$\hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(X_i < c)$$
and derive its asymptotic distribution. Note that $\mathbb{I}(X_i < c)$ is an indicator function such that
$$\mathbb{I}(X_i < c) = \begin{cases} 1 & \text{if } X_i < c \\ 0 & \text{otherwise} \end{cases}$$
3. Suppose that $c = \mu = 0$. Which of the two estimators would you prefer based on your asymptotic approximations above?

Part II - 203B

Question II-1

Suppose that

$$y = X\beta + \varepsilon$$

where X is a nonstochastic $n \times k$ matrix with full column rank, $E[\varepsilon] = 0$, and $E[\varepsilon\varepsilon'] = \sigma^2 I_n$. Let B and C be nonstochastic $n \times k$ matrices with $BX = CX = I$, and

$$\widehat{\beta} = By$$

and

$$\widetilde{\beta} = Cy$$

Let B be chosen so that $\widehat{\beta}$ is the best linear unbiased estimator for β. Prove that

$$E\left[\left(\widetilde{\beta} - \widehat{\beta}\right)\left(\widehat{\beta} - E\left[\widehat{\beta}\right]\right)'\right] = 0$$

Question II-2

Suppose that $X_1, \ldots, X_n$ are iid $N(\theta_1, \theta_2)$. Calculate the Fisher information for (θ_1, θ_2) from $(X_1, \ldots, X_n)$. Let

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$$

where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Is the variance of s^2 equal to the inverse of the Fisher information for θ_2? (You are required to state the variance of s^2, but you do not have to derive it. You are required to make the comparison correctly, i.e., show that one of them is strictly bigger than the other or they are equal to each other.)

Question II-3

Suppose we want to estimate a Cobb-Douglas production function

$$y_i = \alpha + \beta_L \cdot l_i + \beta_K \cdot k_i + \varepsilon_i, \quad i = 1, \ldots, n,$$

where

$$y_i = \log(\text{output}), \quad l_i = \log(\text{labor}), \quad k_i = \log(\text{capital})$$

of the i^{th} firm. You want to estimate β_L by OLS.

Suppose that the model satisfies the assumptions of Classical Linear Regression Model II (deterministic regressor with full column rank, zero mean normal independent errors with same variances). Suppose that you estimated $(\alpha, \beta_L, \beta_K)$ by OLS. (You purchased a better software.) Your computer reported

variable	estimated coefficient	estimated standard error	t ratio
Constant	1.000	0.4	2.5
log(labor)	0.6000	0.3	2.0
log(capital)	0.3700	0.2	1.85

Your computer also reported the estimated variance covariance matrix

	Constant	log(labor)	log(capital)
Constant	0.16	$\cdot$	$\cdot$
log(labor)	-0.06	0.09	$\cdot$
log(capital)	0.032	-0.018	0.04

The number of observations n is equal to 27, and the sum of squard residuals is equal to 0.18. When you answer the questions below, you may assume that the critical values from the t-distribution are identical to those from the standard normal distribution. What is the 95% confidence interval of $\beta_L + \beta_K$? You do not need to take the square roots, but you should finish every other algebra, i.e., addition/subtraction/division/multiplication.

Question II-4

Suppose that

$$y_i^* = \beta \cdot x_i + \varepsilon_i,$$

Our data consist of (y_i, x_i, z_i, D_i) $i = 1, \ldots, n$, where (x_i, z_i) is nonstochastic and

$$y_i \equiv \begin{cases} y_i^* & \text{if } D_i = 1 \\ 0 & \text{if } D_i = 0 \end{cases},$$

where

$$D_i \equiv \begin{cases} 1 & \text{if } \gamma \cdot z_i + u_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

We know that (x_i, z_i) is independent of (ε_i, u_i) as well as

$$\begin{pmatrix} \varepsilon_i \\ u_i \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_\varepsilon^2 & 0 \\ 0 & \sigma_u^2 \end{bmatrix} \right)$$

You decided to regress y_i on x_i in the subsample where $D_i = 1$. Would you get a consistent estimator of β? Why or why not?

Part III - 203C

Question III-1

Suppose that $\{Y_i\}_{i=1}^n$ $(n > 2)$ form a random sample from a normal distribution with unknown mean μ and variance σ^2. Consider the hypotheses $H_0 : \sigma^2 = \sigma_o^2$ against $H_1 : \sigma^2 \neq \sigma_o^2$ where σ_o^2 is a known positive real number. Construct the likelihood ratio test and show that its critical region is

$$\left\{ \{Y_i\}_{i=1}^n : \sum_{i=1}^n (Y_i - \overline{Y}_n)^2 \geq c_1 \text{ or } \sum_{i=1}^n (Y_i - \overline{Y}_n)^2 \leq c_2 \right\}$$

for some positive constants c_1 and c_2, where $\overline{Y}_n = n^{-1} \sum_{i=1}^n Y_i$.

Question III-2

Suppose that $\{Y_t\}$ is an auto-regressive process, i.e.

$$Y_t = \rho_o Y_{t-1} + u_t,$$

where $|\rho_o| < 1$ and $u_t \sim i.i.d.(0, \sigma_u^2)$ with $E[u_t^4] < \infty$.

1. **(a)** Find the explicit form of the long-run variance ω_Y^2 of $\{Y_t\}$.

   (b) Derive the asymptotic distribution of the OLS estimator

   $$\widehat{\rho}_n = \frac{\sum_{t=2}^n Y_t Y_{t-1}}{\sum_{t=1}^n Y_t^2}.$$

   (c) Let $K > 2$ be any fixed positive integer. Is ρ_o uniquely identified by

   $$E\Big[\ (Y_t - \rho_o Y_{t-1}) Y_{t-K} \ \Big] = 0?$$

   (d) Consider the following instrumental variable (IV) estimator of ρ_o

   $$\widehat{\rho}_{n,K}^{iv} = \frac{\sum_{t=K+1}^n Y_t Y_{t-K}}{\sum_{t=K+1}^n Y_{t-1} Y_{t-K}}.$$

   Derive the asymptotic distribution of $\widehat{\rho}_{n,K}^{iv}$.

   (e) Is the IV estimator $\widehat{\rho}_{n,K}^{iv}$ a consistent estimator of ρ_o? When your answer is "yes", (i) compare the asymptotic variance of $\widehat{\rho}_{n,K}^{iv}$ with that of $\widehat{\rho}_n$; (ii) compare the asymptotic variances of $\widehat{\rho}_{n,K_1}^{iv}$ and $\widehat{\rho}_{n,K_2}^{iv}$ with $K_1 \neq K_2$; (iii) explain your findings in (ii) and (iii).

Question III-3

Suppose that Y_t is generated from the following model

$$Y_t = \theta_o + u_t$$

where $u_t = \rho_o u_{t-1} + v_t$ with $v_t \sim i.i.d.(0,1)$ and $E[v_t^4] < \infty$.

1. **(a)** Suppose that $|\rho_o| < 1$. Find the asymptotic distribution of $\widehat{\theta}_n = n^{-1} \sum_{t=1}^n Y_t$.

   (b) Suppose that $|\rho_o| < 1$ and $\theta_o = 0$. We use the data $\{Y_t\}_{t=1}^n$ to fit the simple AR(1) model

$$Y_t = \widehat{\alpha}_n Y_{t-1} + \widehat{v}_t, \text{ where } \widehat{v}_t = Y_t - \widehat{\alpha}_n Y_{t-1}$$
$$\widehat{\alpha}_n = \frac{\sum_{t=2}^n (Y_t - \overline{Y}_n) Y_{t-1}}{\sum_{t=1}^n (Y_t - \overline{Y}_n)^2} \text{ and } \overline{Y}_n = n^{-1} \sum_{t=1}^n Y_t.$$

   Find the probability limit α^* of $\widehat{\alpha}_n$ and then derive the asymptotic distribution of $\widehat{\alpha}_n$. (Hint: you are supposed to show that $\widehat{\alpha}_n - \alpha^*$ scaled by some sequence which diverges with sample size n convergence in distribution to some non-degenerated random variable.)

   (c) If $\rho_o = 1$ and $\theta_o = 0$, will your answers in (a) and (b) be changed? Please be specific if your answer(s) is (are) different.

   In the rest of this problem, i.e. in (d), (e), (f) and (g), we maintain the assumption that $|\rho_o| < 1$.

   (d) Suppose that researcher A suggests to form 1-period ahead forecast of Y_t using $\widehat{\theta}_n$ for all t. Then the within sample mean square predicting error is

$$MSPE_{1,n} = \frac{1}{n} \sum_{t=1}^n (Y_t - \widehat{\theta}_n)^2.$$

   Derive the probability limit of $MSPE_{1,n}$.

   (e) Suppose that researcher B suggests to form 1-period ahead forecast of Y_t using $\widehat{\alpha}_n$ for all t. Then the within sample mean square predicting error is

$$MSPE_{2,n} = \frac{1}{n} \sum_{t=2}^n (Y_t - \widehat{\alpha}_n Y_{t-1})^2.$$

   Derive the probability limit of $MSPE_{2,n}$.

   (f) Compare the results you get in (d) and (e) and explain your findings.

Some Useful Theorems and Lemmas

Theorem 1 (Martingale Convergence Theorem) *Let $\{(X_t, \mathcal{F}_t)\}_{t \in \mathbb{Z}_+}$ be a martingale in L^2. If $\sup_t E\left[|X_t|^2\right] < \infty$, then $X_n \to X_\infty$ almost surely, where X_∞ is some element in L^2.*

Theorem 2 (Martingale CLT) *Let* $\{X_{t,n}, \mathcal{F}_{t,n}\}$ *be a martingale difference array such that* $E[|X_{t,n}|^{2+\delta}] < \Delta < \infty$ *for some* $\delta > 0$ *and for all* t *and* n. *If* $\overline{\sigma}_n^2 > \delta_1 > 0$ *for all* n *sufficiently large and* $\frac{1}{n}\sum_{t=1}^n X_{t,n}^2 - \overline{\sigma}_n^2 \to_p 0$, *then* $n^{\frac{1}{2}}\overline{X}_n/\overline{\sigma}_n \to_d N(0,1)$.

Theorem 3 (LLN of Sample Variance) *Suppose that* Z_t *is i.i.d. with mean zero and* $E[Z_0^2] = \sigma_Z^2 < \infty$. *Let* $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, *where* φ_k *is a sequence of real numbers with* $\sum_{k=0}^\infty k\varphi_k^2 < \infty$. *Then*

$$\frac{1}{n}\sum_{t=1}^n X_t X_{t-h} \to_p \Gamma_X(h) = E\left[X_t X_{t-h}\right]. \tag{1}$$

Theorem 4 (Donsker) *Let* $\{u_t\}$ *be a sequence of random variables generated by* $u_t = \sum_{k=0}^\infty \varphi_k \varepsilon_{t-k} = \varphi(L)\varepsilon_t$, *where* $\{\varepsilon_t\} \sim iid\ (0, \sigma_\varepsilon^2)$ *with finite fourth moment and* $\{\varphi_k\}$ *is a sequence of constants with* $\sum_{k=0}^\infty k\,|\varphi_k| < \infty$. *Then* $B_{u,n}(\cdot) = n^{-\frac{1}{2}}\sum_{t=1}^{[n\cdot]} u_t \to_d \lambda B(\cdot)$, *where* $\lambda = \sigma_\varepsilon\varphi(1)$.