First Year Quantitative Comp Exam
Fall 2012

Instruction: There are three parts. Answer every question in every part.

Part I - 203A

1. A random variable X attains the value 1 with probability 1/10, the value 2 with probability 5/10, and the value 3 with probability 4/10. Another random variable, Y, is distributed, given $X = x$, with the conditional density

$$f_{Y|X=x}(y) = \begin{cases} x \ \exp(-xy) & \text{if} \quad 0 < y \\ 0 & \text{otherwise} \end{cases}$$

   (a) Let μ_Y denote the expectation of Y. Calculate μ_Y.

   (b) Calculate $Cov(X, Y)$. Are X and Y independently distributed? Explain.

   (c) Calculate $E[X|\, Y = \mu_Y]$.

   (d) Calculate $\Pr(X > Y)$.

2. The value of a random variable Y^* is determined by a random vector X and a random variable ε according to the relationship

$$Y^* = \alpha + \beta X + \gamma\varepsilon$$

   where α, β, and γ are parameters of unknown values, $X = 0$ with probability 1/10 and $X = 1$ with probability 9/10, and ε conditional on $X = x$ is distributed $N(x, \sigma^2)$. The random variable Y is determined by

$$Y = \begin{cases} Y^* & \text{if} \quad Y^* > 0 \\ 0 & \text{otherwise} \end{cases}$$

   Suppose that only (Y, X) is observed.

   (a) Obtain an expression for $\Pr(Y = 0)$ in terms of the unknown parameters.

   (b) Determine what parameters, or functions of parameters, if any, are identified. Explain. How would your answer change if it were known that $\beta = 1$? Explain

   (c) Suppose that $\gamma = 1$ and ε is distributed independently of X with an unknown, everywhere positive density f_ε. Analyze the identification of α, β, and f_ε.

(d) Suppose that $\gamma = 1$, $\beta = 0$, and that ε is distributed $N(0, \sigma^2)$, independently of X. Let

$$Z = \begin{cases} 1 & \text{if} \quad Y^* > 0 \\ 0 & \quad\text{otherwise} \end{cases}$$

and

$$W = \begin{cases} 1 & \text{if} \quad Y^* > 2 \\ 0 & \quad\text{otherwise} \end{cases}$$

What is the joint probability density of (Z, W)? Explain.

Part II - 203B

1. Let

$$y_i = x_i\beta + \varepsilon_i$$

such that $E[z_i\varepsilon_i] = E[z_i]E[\varepsilon_i]$ and $E[z_i x_i] \neq E[z_i]E[x_i]$. We do **not** assume that $E[\varepsilon_i] = 0$. We observe (y_i, x_i, z_i), $i = 1, 2, \ldots, n$. Is β identified? Why or why not? If it is identified, propose a consistent estimator of β under the assumption that (y_i, x_i, z_i) are iid, and prove that your estimator is consistent.

2. Let

$$y_i = x_{i1}\beta_1 + x_{i2}\beta_2 + \varepsilon_i$$

such that $(x_{i1}, x_{i2})'$ $i = 1, 2, \ldots, n$ are **non**stochastic, and ε_i are iid with mean zero and variance 1. It is known that

$$\begin{bmatrix} \sum_{i=1}^n x_{i1}^2 & \sum_{i=1}^n x_{i1}x_{i2} \\ \sum_{i=1}^n x_{i1}x_{i2} & \sum_{i=1}^n x_{i2}^2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

Let $\widehat{\theta}$ denote the OLS estimator when y_i is regressed on $x_{i1} - \frac{1}{2}x_{i2}$. What is the mean of $\widehat{\theta}$? What is the variance of $\widehat{\theta}$?

3. Suppose that

$$y_i = x_{i1}^*\beta_1 + x_{i2}^*\beta_2 + \varepsilon_i$$

We observe (y_i, x_{i1}, x_{i2}), where $x_{i1} = x_{i1}^* + u_{i1}$ and $x_{i2} = x_{i2}^* + u_{i2}$. Let $\left(\widehat{\beta}_1, \widehat{\beta}_2\right)'$ denote the OLS estimator when y_i is regressed on (x_{i1}, x_{i2}). You may assume that the true value of (β_1, β_2) is equal to $(1, 1)$; make an explicit statement if you do impose such an assumption.

   (a) Assume that $x_{i1}^*, x_{i2}^*, \varepsilon_i, u_{i1}, u_{i2}$ are independent of each other. Also assume that they all have mean equal to zero and variance equal to one. What is the probability limit of $\widehat{\beta}_1$? Is $\operatorname{plim}\widehat{\beta}_1$ smaller than, equal to, or larger than β_1?

   (b) Assume now that $x_{i1}^*, x_{i2}^*, \varepsilon_i, u_{i1}$ all have mean equal to zero and variance equal to one. As for u_{i2}, we assume that it is identically equal to zero, i.e., there is no measurement error. Also assume that (i) the vectors (x_{i1}^*, x_{i2}^*) and (ε_i, u_{i1}) are independent of each other; (ii) ε_i, u_{i1} are independent of each other; (iii) the covariance between x_{i1}^* and x_{i2}^* is ρ. What is the probability limit of $\widehat{\beta}_2$? Is $\operatorname{plim}\widehat{\beta}_2$ smaller than, equal to, or larger than β_2?

Part III - 203C

1. Suppose X has the density function of the form

$$f(x;\theta) = \begin{cases} \theta^x(1-\theta)^{1-x}, & \text{if } x = 0, 1 \\ 0 & \textit{otherwise} \end{cases}.$$

We test $H_0 : \theta = \frac{1}{2}$ against $H_1 : \theta < \frac{1}{2}$ by taking a random (*i.i.d.*) sample $\{X_i\}_{i=1}^{5}$ with sample size $n = 5$ and rejecting H_0 if $Y_n = \sum_{i=1}^{5} X_i$ is observed to be less than or equal to a constant c.

   (a) Show that this is a uniform most powerful test.

   (b) Find the significance level when $c = 1$.

   (c) Find the significance level when $c = 0$.

2. Consider the following simple AR(1) model

$$Y_t = \alpha_o Y_{t-1} + u_t, \text{ with } u_t \sim i.i.d. \ (0, \sigma_u^2)$$

where the unknown parameter α_o satisfies $|\alpha| < 1$.

   (a) Derive the limiting distribution of the OLS estimate $\widehat{\alpha}_n$ defined as

$$\widehat{\alpha}_n = \frac{\sum_{t=1}^{n} Y_t Y_{t-1}}{\sum_{t=1}^{n} Y_{t-1}^2}. \tag{1}$$

   (b) Find a consistent estimate of the asymptotic variance of the OLS estimate $\widehat{\alpha}_n$ and construct a consistent test for the hypothesis $H_0 : \alpha_o = 0$.

   Now suppose that y_t is observable only when t is a odd number and you have n such observations $\{Y_{2t-1}\}_{t=1}^{n}$. In this case, the OLS estimate becomes

$$\widehat{\alpha}_n^* = \frac{\sum_{t=2}^{n} Y_{2t-1} Y_{2t-3}}{\sum_{t=1}^{n} Y_{2t-1}^2}. \tag{2}$$

   (c) Is $\widehat{\alpha}_n^*$ a consistent estimate of α? Derive the probability limit of $\widehat{\alpha}_n^*$. Based on the probability limit of $\widehat{\alpha}_n^*$, find a consistent estimate of α_o.

   (d) Derive the limiting distribution of the consistent estimate you proposed in (c).

   (e) Find a consistent estimate of the asymptotic variance of the consistent estimate of α_o you proposed in (c). Construct a consistent test for the hypothesis $H_0 : \alpha_o = 0$.

3. The trend regression equation

$$Y_t = \alpha_o \rho^t + u_t, \ t = 1, ..., n \tag{3}$$

models the scalar observed time series Y_t in terms of the exponential trend ρ^t, where $\rho > 1$ is known, α_o is an unknown coefficient to be estimated and u_t is *i.i.d.* normal $(0, \sigma_u^2)$ with $\sigma_u^2 > 0$.

(a) The model (3) is fitted by linear least squares regression giving coefficient estimate

$$\widehat{\alpha}_n = \frac{\sum_{t=1}^n \rho^t Y_t}{\sum_{t=1}^n \rho^{2t}}.$$

Show that $\widehat{\alpha}_n$ is consistent and find its limiting distribution.

(b) Using data generated from (3), the first order autoregression $Y_t = \widehat{\beta}_n Y_{t-1} + \widehat{v}_t$ is fitted, i.e.

$$\widehat{\beta}_n = \frac{\sum_{t=1}^n Y_t Y_{t-1}}{\sum_{t=1}^n Y_{t-1}^2}$$

and $\widehat{v}_t$ is the fitted residual. Find the asymptotic behavior of $\widehat{\beta}_n$ as $n \to \infty$.

4. Suppose that

$$Y_t = X_t \beta + u_t + c \sum_{s=0}^{t-1} u_s \tag{4}$$

where $u_t \sim i.i.d.\ (0,1)$, c and β are some unknown constants and

$$X_t = X_{t-1} + \varepsilon_t$$

where ε_t is an $i.i.d.\ (0,1)$ sequence independent of u_s for all t and s, and $X_0 = 0$.

(a) Let $\Delta X_t = X_t - X_{t-1}$ and $\Delta Y_t = Y_t - Y_{t-1}$, what's the limiting distribution of the following estimate?

$$\widehat{\beta}_n^* = \sum_{t=2}^n \Delta X_t \Delta Y_t / \sum_{t=2}^n \Delta X_t^2.$$

(b) Based on the estimate $\widehat{\beta}_n^*$, construct a consistent estimate of c in (4).