UCLA Economics Dept
Fall 2000
Quantitative Methods
4 hours

Quantitative Exam, Fall '00

There are three parts to the exam, with two questions for each part, for a total of six questions. You should answer all six questions. Use a separate blue book for each of the three parts.

Part I

1. Let the random variable X have a normal distribution with mean μ and variance μ^2, with $\mu > 0$.

   (a) Find an unbiased estimator for μ.

   (b) Find the Cramer-Rao bound for unbiased estimators.

   (c) Is there an unbiased estimator with variance equal to the Cramer-Rao bound?

   (d) Find the maximum likelihood estimator.

   (e) Is the maximum likelihood estimator unbiased?

   (f) Suppose you have a random sample $x_1, \ldots, x_N$ of size $N = 100$ from this distribution with $\sum_{i=1}^{N} x_i = 110$, and $\sum_{i=1}^{N} x_i^2 = 250$. Calculate the maximum likelihood estimate and its approximate variance.

2. Let the marginal distribution of a random variable X be binomial with $N = 1$ and $p = 1/2$. Conditional on X, the random variable Y has an exponential distribution with mean $(1 + X) \cdot \lambda$.

   (a) Find the marginal pdf of Y.

   (b) Find the conditional probability that $X = 1$ given $Y = y$.

   (c) Suppose you have a random sample of size N from this joint distribution. What is the maximum likelihood estimator and what is its large sample variance?

   (d) Suppose you only observe the values of Y. What is the maximum likelihood estimator in that case and what is its large sample variance? How does it compare to the variance in the previous question?

   (e) Suppose $\sum X_i = 50$, $N = 100$, $\sum X_i Y_i = 200$, and $\sum (1 - X_i) Y_i = 110$. Test the null hypothesis that $\lambda = 2$ at the 10% level using a likelihood ratio test.

Part II

PROBLEM 1:

Suppose that you have n independent observations on (y_i, x_i) from the GCR model:

$$y_i = x_i\beta + \varepsilon_i$$

where $Var(\varepsilon_i) = \sigma_i^2$ and $Cov(\varepsilon_i\varepsilon_j) = 0$.

(a) Show that the OLS estimator of β is consistent. Make sure you list all assumptions and theorems you use in your proof.

(b) Suppose that the errors are jointly normally distributed. How would you test the statistical significance of one of the coefficients in β using the OLS estimate? Make sure you provide all details about the test statistic you would use and its distribution.

(c) Let $\sigma_i^2 = \exp(z_i\gamma)$ where z_i is a subvector of x_i and γ is an unknown parameter vector. Describe the best linear unbiased estimator of β and how you would implement it.

PROBLEM 2:

Let the CR model apply with the usual notation. For each of the following statements, indicate whether it is true or false and justify your answer.

(a) The random variable $t = \hat{\beta}'\hat{\beta}$ is an unbiased estimator of the parameter $\theta = \beta'\beta$. (Here $\hat{\beta}$ is the OLS estimator.)

(b) Since $\hat{y} = Ny$, it follows that $y = N^{-1}\hat{y}$. (Here N is the projection matrix.)

(c) Since $E(\hat{y}) = E(y)$ it follows that the sum of residuals is zero.

(d) If $\hat{\beta}_1$ and $\hat{\beta}_2$ are the first two elements of $\hat{\beta}$, $t_1 = \hat{\beta}_1 + \hat{\beta}_2$ and $t_2 = \hat{\beta}_1 - \hat{\beta}_2$, then $Var(t_1) \geq Var(t_2)$.

PART III.

1. Suppose that X_i, $i = 1, \ldots, n$ are i.i.d., continuously distributed on the open interval $(0, 1)$. Y_i are binary variables with

$$Pr(Y_i = 1 | X_1, \ldots, X_n) = x_i \beta$$

We assume that $\beta \in (0, 1)$.

(a) Write down the likelihood function and log-likelihood function for $y_1, \ldots, y_n$ conditional on $x_1, \ldots, x_n$.

(b) Write down the derivative of the log-likelihood with respect to β.

(c) Suppose you estimate β by any value $\hat{\beta} \in (0, 1)$, which sets the derivative of the log-likelihood equal to zero. (Assume that at least one solution exists and lies in $(0, 1)$.) Show that the solution is unique.

(d) How does the estimator in (c) relate to the maximum likelihood estimator? What are the asymptotic properties of the estimator in (c)?

2. Suppose that $\{x_t\}$ is a strictly stationary, mean-zero process with the AR(2) representation

$$x_t = b_1 x_{t-1} + b_2 x_{t-2} + \epsilon_t,$$

where the ϵ_t are mean zero and i.i.d. over time.

(a) Let $w_t = (x_t, x_{t-1})'$. Show that w_t has a VAR(1) representation.

(b) Now suppose that

$$y_t = d x_t + u_t,$$

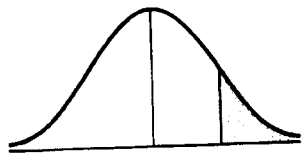
and that

$$\begin{pmatrix} \epsilon_t \\ u_t \end{pmatrix} \overset{\text{i.i.d.}}{\sim} N(0, \Sigma).$$

Let $\hat{d}$ denote the OLS coefficient in a regression of y_t on x_t. Will $\hat{d}$ consistently estimate d? Explain your reasoning fully.

(c) Propose an estimator for d and show that it is consistent. State any additional assumptions you need for consistency to hold.

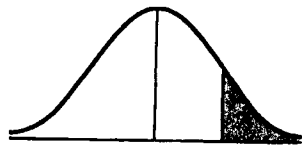
TABLE 1 Normal distribution, right-hand tail probabilities



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002

	.09
1	.4641
6	.4247
7	.3859
0	.3483
6	.3121
0	.2776
3	.2451
7	.2148
4	.1867
5	.1611
1	.1379
0	.1170
3	.0985
8	.0823
4	.0681
1	.0559
5	.0455
5	.0367
1	.0294
9	.0233
8	.0183
6	.0143
3	.0110
7	.0084
6	.0064
9	.0048
7	.0036
7	.0026
0	.0019
4	.0014
0	.0010
7	.0007
5	.0005
4	.0003
3	.0002

TABLE 2 Cutoff points for Student's t distribution, right-hand tail probabilities



df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.611	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1,000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
∞	0.674	0.841	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.091	3.291

TABLE 3 Cutoff points for the chi squared distribution, right-hand tail probabilities

df	.9995	.999	.9975	.995	.990	.975	.950	.900	.750	.500
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.10	0.45
2	0.00	0.00	0.01	0.01	0.02	0.05	0.10	0.21	0.58	1.39
3	0.02	0.02	0.04	0.07	0.11	0.22	0.35	0.58	1.21	2.37
4	0.06	0.09	0.14	0.21	0.30	0.48	0.71	1.06	1.92	3.36
5	0.16	0.21	0.31	0.41	0.55	0.83	1.15	1.61	2.67	4.35
6	0.30	0.38	0.53	0.68	0.87	1.24	1.64	2.20	3.45	5.35
7	0.48	0.60	0.79	0.99	1.24	1.69	2.17	2.83	4.25	6.35
8	0.71	0.86	1.10	1.34	1.65	2.18	2.73	3.49	5.07	7.34
9	0.97	1.15	1.45	1.73	2.09	2.70	3.33	4.17	5.90	8.34
10	1.26	1.48	1.83	2.16	2.56	3.25	3.94	4.87	6.74	9.34
11	1.59	1.83	2.23	2.60	3.05	3.82	4.57	5.58	7.58	10.34
12	1.93	2.21	2.66	3.07	3.57	4.40	5.23	6.30	8.44	11.34
13	2.31	2.62	3.11	3.57	4.11	5.01	5.89	7.04	9.30	12.34
14	2.70	3.04	3.58	4.07	4.66	5.63	6.57	7.79	10.17	13.34
15	3.11	3.48	4.07	4.60	5.23	6.26	7.26	8.55	11.04	14.34
16	3.54	3.94	4.57	5.14	5.81	6.91	7.96	9.31	11.91	15.34
17	3.98	4.42	5.09	5.70	6.41	7.56	8.67	10.09	12.79	16.34
18	4.44	4.90	5.62	6.26	7.01	8.23	9.39	10.86	13.68	17.34
19	4.91	5.41	6.17	6.84	7.63	8.91	10.12	11.65	14.56	18.34
20	5.40	5.92	6.72	7.43	8.26	9.59	10.85	12.44	15.45	19.34
21	5.90	6.45	7.29	8.03	8.90	10.28	11.59	13.24	16.34	20.34
22	6.40	6.98	7.86	8.64	9.54	10.98	12.34	14.04	17.24	21.34
23	6.92	7.53	8.45	9.26	10.20	11.69	13.09	14.85	18.14	22.34
24	7.45	8.08	9.04	9.89	10.86	12.40	13.85	15.66	19.04	23.34
25	7.99	8.65	9.65	10.52	11.52	13.12	14.61	16.47	19.94	24.34
26	8.54	9.22	10.26	11.16	12.20	13.84	15.38	17.29	20.84	25.34
27	9.09	9.80	10.87	11.81	12.88	14.57	16.15	18.11	21.75	26.34
28	9.66	10.39	11.50	12.46	13.56	15.31	16.93	18.94	22.66	27.34
29	10.23	10.99	12.13	13.12	14.26	16.05	17.71	19.77	23.57	28.34
30	10.80	11.59	12.76	13.79	14.95	16.79	18.49	20.60	24.48	29.34
40	16.91	17.92	19.42	20.71	22.16	24.43	26.51	29.05	33.66	39.34
50	23.46	24.67	26.46	27.99	29.71	32.36	34.76	37.69	42.94	49.33
60	30.34	31.74	33.79	35.53	37.48	40.48	43.19	46.46	52.29	59.33
80	44.79	46.52	49.04	51.17	53.54	57.15	60.39	64.28	71.14	79.33
100	59.90	61.92	64.86	67.33	70.06	74.22	77.93	82.36	90.13	99.33

TABLE 3 (continued)

df	.250	.200	.150	.100	.050	.025	.020	.010	.005	.0025	.001	.0005
1	1.32	1.64	2.07	2.71	3.84	5.02	5.41	6.63	7.88	9.14	10.83	12.12
2	2.77	3.22	3.79	4.61	5.99	7.38	7.82	9.21	10.60	11.98	13.82	15.20
3	4.11	4.64	5.32	6.25	7.81	9.35	9.84	11.34	12.84	14.32	16.27	17.73
4	5.39	5.99	6.74	7.78	9.49	11.14	11.67	13.28	14.86	16.42	18.47	20.00
5	6.63	7.29	8.12	9.24	11.07	12.83	13.39	15.09	16.75	18.39	20.51	22.11
6	7.84	8.56	9.45	10.64	12.59	14.45	15.03	16.81	18.55	20.25	22.46	24.10
7	9.04	9.80	10.75	12.02	14.07	16.01	16.62	18.48	20.28	22.04	24.32	26.02
8	10.22	11.03	12.03	13.36	15.51	17.53	18.17	20.09	21.95	23.77	26.12	27.87
9	11.39	12.24	13.29	14.68	16.92	19.02	19.68	21.67	23.59	25.46	27.88	29.67
10	12.55	13.44	14.53	15.99	18.31	20.48	21.16	23.21	25.19	27.11	29.59	31.42
11	13.70	14.63	15.77	17.28	19.68	21.92	22.62	24.72	26.76	28.73	31.26	33.14
12	14.85	15.81	16.99	18.55	21.03	23.34	24.05	26.22	28.30	30.32	32.91	34.82
13	15.98	16.98	18.20	19.81	22.36	24.74	25.47	27.69	29.82	31.88	34.53	36.48
14	17.12	18.15	19.41	21.06	23.68	26.12	26.87	29.14	31.32	33.43	36.12	38.11
15	18.25	19.31	20.60	22.31	25.00	27.49	28.26	30.58	32.80	34.95	37.70	39.72
16	19.37	20.47	21.79	23.54	26.30	28.85	29.63	32.00	34.27	36.46	39.25	41.31
17	20.49	21.61	22.98	24.77	27.59	30.19	31.00	33.41	35.72	37.95	40.79	42.88
18	21.60	22.76	24.16	25.99	28.87	31.53	32.35	34.81	37.16	39.42	42.31	44.43
19	22.72	23.90	25.33	27.20	30.14	32.85	33.69	36.19	38.58	40.88	43.82	45.97
20	23.83	25.04	26.50	28.41	31.41	34.17	35.02	37.57	40.00	42.34	45.31	47.50
21	24.93	26.17	27.66	29.62	32.67	35.48	36.34	38.93	41.40	43.78	46.80	49.01
22	26.04	27.30	28.82	30.81	33.92	36.78	37.66	40.29	42.80	45.20	48.27	50.51
23	27.14	28.43	29.98	32.01	35.17	38.08	38.97	41.64	44.18	46.62	49.73	52.00
24	28.24	29.55	31.13	33.20	36.42	39.36	40.27	42.98	45.56	48.03	51.18	53.48
25	29.34	30.68	32.28	34.38	37.65	40.65	41.57	44.31	46.93	49.44	52.62	54.95
26	30.43	31.79	33.43	35.56	38.89	41.92	42.86	45.64	48.29	50.83	54.05	56.41
27	31.53	32.91	34.57	36.74	40.11	43.19	44.14	46.96	49.64	52.22	55.48	57.86
28	32.62	34.03	35.71	37.92	41.34	44.46	45.42	48.28	50.99	53.59	56.89	59.30
29	33.71	35.14	36.85	39.09	42.56	45.72	46.69	49.59	52.34	54.97	58.30	60.73
30	34.80	36.25	37.99	40.26	43.77	46.98	47.96	50.89	53.67	56.33	59.70	62.16
40	45.62	47.27	49.24	51.81	55.76	59.34	60.44	63.69	66.77	69.70	73.40	76.09
50	56.33	58.16	60.35	63.17	67.50	71.42	72.61	76.15	79.49	82.66	86.66	89.56
60	66.98	68.97	71.34	74.40	79.08	83.30	84.58	88.38	91.95	95.34	99.61	102.7
80	88.13	90.41	93.11	96.58	101.9	106.6	108.1	112.3	116.3	120.1	124.8	128.3
100	109.1	111.7	114.7	118.5	124.3	129.6	131.1	135.8	140.2	144.3	149.4	153.2