Spring 2016

1. **Equilibrium with Uncertainty**

Consider a two-period exchange economy with one good and two consumers. There are two states of the world s_1 and s_2, which are equally likely. At state s_1, consumer 1 is endowed with 2 goods and consumer 2 is endowed with 1 good At s_2, consumer 2 is endowed with 2 goods and consumer 1 with 1 good. In the first period, a state of the world realizes and nothing else happens. Consumption occurs only in the 2nd period. Each consumer is an expected utility maximizer with a log (Bernoulli) utility function $u(x) = \log x$. Thus consumer i's expected utility from consumption plan $x_i = (x_{i,1}, x_{i,2})$ is given by $0.5 \log x_{i,1} + 0.5 \log x_{i,2}$ ($x_{i,s}$ is consumer i's consumption at state s). Answer the following questions.

(a) Find all the Pareto efficient allocations.

(b) Find an Arrow-Debreu equilibrium (remember that it is just a usual Walrasian equilibrium where goods are state-contingent goods).

Assume that the following two financial assets are available for trading in the first period for the rest of the questions. Asset A pays out 1 (unit of account) in both states in the 2nd period. Asset B pays 2 at state s_1 and 3 at state s_2. Let q_k be the price of Asset k for $k = A, B$.

(c) Show that there is an opportunity for arbitrage when $(q_A, q_B) = (3, 5)$.

(d) Find asset prices (q_A, q_B) given which there is no arbitrage opportunity.

(e) Find a financial equilibrium/Radner equilibrium (consumption, asset holding, prices of the assets and state-contingent goods) that implements the same allocation as the Arrow-Debreu equilibrium allocation in (b).

Answer for Q1

(a) (2 pts.) Since the consumers have the same Bernoulli utility function and their preferences are homothetic, the set of Pareto efficient allocations coincide with the diagonal line of the Edgeworth box: $x_1 = (3\alpha, 3\alpha)$, $x_2 = (3(1-\alpha), 3(1-\alpha))$, where $\alpha \in [0,1]$

(b) (2.5 pts.) The equilibrium allocation must be Pareto efficient by the first welfare theorem. So the equilibrium price ratio $\frac{p_1^*}{p_2^*}$ must be 1 by the tangency condition. In equilibrium, consumer 1 sells 0.5 unit of good at state s_1 to consumer 2 and buys 0.5 unit of good at state s_2 from consumer 2 so that his equilibrium consumption is 1.5 at each state. The equilibrium allocation is $x_i^* = (1.5, 1.5)$ for $i = 1, 2$ with any equilibrium price $p^* = (p_1^*, p_2^*)$ such that $p_1^* = p_2^* > 0$.

(c) (1.5 pts.) Sell 5 units of Asset A and buy 3 units of Asset B. Since they are of equal value (= 15), this trade is feasible without incurring any cost in the first period. Then one can get 1 in state s_1 and 4 in state s_2 for free.

(d) (1.5 pts.) There is no arbitrage with asset price (q_A, q_B) if and only if there exists $(\lambda_1, \lambda_2) \gg 0$ (state price) that satisfy the following:

$$q_A = \lambda_1 + \lambda_2, \; q_B = 2\lambda_1 + 3\lambda_2$$

For example, pick $(\lambda_1, \lambda_2) = (1, 1)$. Then $(q_A, q_B) = (2, 5)$ satisfies this condition.

(e) (2.5 pts.) Set the asset prices to $(q_A, q_B) = (2, 5)$ (any no arbitrage price would work). Since consumer 1 needs money to purchase goods at state 2, consumer 1 needs to buy asset B and sells Asset A. So suppose that consumer 1 buys 2 unit of asset B and sells 5 unit of asset A (there are many other quantities that would work). Then consumer 1 will owe 1 unit of account at state s_1 and receive 1unit of account at state s_2. Of course consumer 2 takes the opposite position and will receive 1 at state s_1 and owe 1 at state s_2.To implement the A-D equilibrium allocation in (b), the price of the good must be 2 at both states so that 1 unit of account is worth 0.5 unit of good at both states. We know that the budget set is the same given $p_1^* = p_2^*$ for A-D equilibrium and given $(q_A, q_B) = (2, 5)$, $p_1' = p_2' = 2$ for financial equilibrium. Hence, given those prices, it is indeed optimal for each consumer to consume 1.5 units of each good at both states. Thus $(x_1^*, x_2^*, (q_A, q_B), p_1', p_2') = ((1.5, 1.5), (1.5, 1.5), (2, 5), 2, 2)$ is a financial equilibrium (there are many other equilibria that implement the same allocation).

2. **Equilibrium with Indivisible Goods**

We usually assume that goods are divisible: a consumer can consume any positive amount of any good. But what would happen if goods are *indivisible*? Many goods are indeed indivisible in real world. For example, you can buy 1 laptop or 2 laptops, but not 1.2 laptop. Here we consider a simple two good-two person pure exchange economy where goods are indivisible (Formally the set of feasible consumption vectors for consumer i is $X_i = \{(k_1, k_2) \,| k_1, k_2 \in \mathbb{N}\}$ and consumer i's endowment e_i is a pair of natural numbers). Assume that consumers' utility functions are linear and strongly increasing in both goods, i.e. $u_i(x) = \alpha_i x_{i,1} + \beta_i x_{i,2}$ with some $(\alpha_i, \beta_i) \gg 0$.

(a) Write down the conditions for (x_1^*, x_2^*, p^*) to be a Walrasian equilibrium in this economy.

(b) Explain why every Pareto-efficient allocation must be on the boundary of the Edgeworth box when $\frac{\alpha_1}{\beta_1} \neq \frac{\alpha_2}{\beta_2}$ (For question (b)-(d), a graphical argument would suffice).

(c) Does there always exist a Walrasian equilibrium in this economy? (Hint: consider using a Pareto-efficient allocation).

(d) Show by an example that there may exist a Walrasian equilibrium in which the equilibrium allocation is not on the boundary of the Edgeworth box (hence is not Pareto-efficient by (b)).

(e) Suppose that good 1 is indivisible, but good 2 is divisible as usual. Does the first welfare theorem hold in this case? If you think so, provide a full proof. If not, find a counter example.

Answer for Q2

(a) (2 pts.) (x_1^*, x_2^*, p^*) is a Walrasian equilibrium if (1) $x_i^* \in X_i$ maximizes consumer i's utility given the budget set, i.e. $p^* x_i^* \leq p^* e_i$ and $x_i^* \succeq_i x_i$ for any $x_i \in X_i$ such that $p^* x_i' \leq p^* e_i$, and (2) the market is clear: $x_1^* + x_2^* = e_1 + e_2$ or $x_1^* + x_2^* \leq e_1 + e_2$.

(b) (2 pts.) Assume that $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$ without loss of generality. Take any interior allocation (x_1, x_2) in the Edgeworth box. Consider a nearby allocation $(x_1', x_2') = (x_1 + (\triangle_1, -\triangle_2), x_2 + (-\triangle_1, \triangle_2))$, where $\triangle_i > 0$ is small and satisfies $\frac{\alpha_1}{\beta_1} > \frac{\triangle_2}{\triangle_1} > \frac{\alpha_2}{\beta_2}$. Then both consumers are better off at (x_1', x_2'). (Note: Here I implicitly assume that you would apply the familiar notion of Pareto-efficiency (with respect to divisible goods). But you can define Pareto-efficiently with indivisiblility. Then you have a larger set of Pareto-efficient allocations and Pareto-efficient allocations may not be on the boundary. It is a perfectly right answer to mention this.)

(c) (2 pts.) This is case by case. Again assume that $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$. Then one Pareto-efficient allocation would be that consumer 1 consumes $x_1' = (r_1, 0)$ (consume all good 1 and consume no good 2) and consume 2 consumes $x_2' = (0, r_2)$, where $r_\ell = e_{1,\ell} + e_{2,\ell}$ for $\ell = 1, 2$. Suppose that x_i' is at least as good as e_i for $i = 1, 2$. Draw a straight line between x' and e in the Edgeworth box and consider any price vector p^* to make this line the budget line. Then (x', p^*) must be a Walrasian equilibrium. Since consumer i's utility must be weakly increasing as i's consumption moves from e_i to x_i' (because of linear utility functions) and consumer i's consumption cannot go beyond x_i' along this budget line, x_i' is indeed the optimal choice for consumer i for $i = 1, 2$. The market clearing condition is trivially satisfied in the Edgeworth box.

In general, there may not exist a Warlasian equilibrium. For example, suppose that $e_1 = (1, 1)$ and $e_2 = (9, 1)$. Also assume that $(\alpha_1, \beta_1) = (2, 1)$ and $(\alpha_2, \beta_2) = (2, 3)$. Note that the allocation $(x_1, x_2) = ((2, 0), (8, 2))$ is Pareto-improving relative to (e_1, e_2). If $\frac{p_1}{p_2} > 1$, then consumer 1 demands at least one unit of good 2 (do not sell good 2) and consumer 2 consumes at least two units of good 2. Hence there is an excess demand for good 2. If $\frac{p_1}{p_2} \leqq 1$, then consumer 1 demands at least two units of good 1 and consumer 2 either demand at least 9 units of good 1 (when $\frac{p_1}{p_2} < \frac{2}{3}$) or demand more than 2 units of good 2 (when $\frac{p_1}{p_2} \geqq \frac{2}{3}$). Again there is an excess demand for good 1 or good 2. So there does not exist any Walrasian equilibrium.

(d) (2 pts.) The following example shows that there may exist a Walrasian equilibrium with an interior allocation, which is not Pareto-efficient (even with respect to the more permissible definition based on indivisibility). Suppose that consumer 1's initial endowment is $(2, 1)$ and consumer 2's initial endowment is $(1, 2)$. Also suppose that consumer 1's utility function is $u_1(x_{1,1}, x_{1,2}) = x_{1,1} + x_{1,2}$ and consumer 2's utility function is $u_2(x_{2,1}, x_{2,2}) = (1 + \varepsilon) x_{2,1} + x_{2,2}$.

Given $(p_1^*, p_2^*) = (1 + 2\varepsilon, 1)$, where ε is small, the optimal consumption bundles for consumer 1 are $(2,1)$, $(1,2)$, and $(0,3)$ and the optimal consumption bundle for consumer 2 is $(1,2)$ and $(0,3)$. Hence it is an equilibrium for both consumers to consume their endowments without any trade given $(p_1^*, p_2^*) = (1 + 2\varepsilon, 1)$. However, since consumer 1 is indifferent between $(2,1)$ and $(1,2)$, consumer 2 strictly prefers $(2,1)$ to $(1,2)$, a feasible allocation $(x_1, x_2) = ((1,2), (2,1))$ is more efficient than $((2,1), (1,2))$.

(e) (2 pts.) Now the local nonsatiation assumption is satisfied because good 2 is divisible and each consumer always prefers more consumption of good 2. So the standard proof of the first welfare theorem works, which is as follows.

Let (x^*, p^*) be a Walrasian equilibrium. Suppose that it is not Pareto-efficient. Then there is a different feasible allocation $x' = (x_1',\ x_2')$ such that $x_1' \succeq_1 x_1^*$ and $x_2 \succeq_2 x_2^*$ with one consumer strictly prefers x_i' more.

Suppose $x_1' \succ_1 x_1^*$ without loss of generality. Since x_1^* is an optimal choice for consumer 1, it must be the case that consumer 1 cannot afford x_1', i.e. $p^* \cdot x_1' > p^* \cdot e_1$. For consumer 2, it must be the case that $p^* \cdot x_2' \geq p^* \cdot e_2$, otherwise consumer 2 can purchase x_2' and add a little bit more of good 2 instead of consuming x_2^* and get strictly better off. So we have

$$p^* \cdot (x_1' + x_2') > p^* \cdot (e_1 + e_2).$$

On the other hand, since the equilibrium price p^* must be strictly positive and $x' \geq 0$ is feasible (i.e., $x_1' + x_2' \leq e_1 + e_2$), we have $p^* \cdot (x_1' + x_2') \leq p^* \cdot (e_1 + e_2)$. This is a contradiction.

Repeated Games ROW and COL play the following asymmetric version of Prisoner's Dilemma infinitely often. They discount future payoffs at the constant rate $\delta > 0$.

	C	D
C	(4,4)	(-2,5)
D	(2,0)	(1,1)

(a) Find the smallest discount factor for which there is a SGPE in which (C,C) is played every period.

(b) Find the smallest discount factor for which there is a SGPE in which play alternates (C,C), (D,C), (C,C), (D,C), ...

(c) Find the smallest discount factor for which there is a SGPE in which play alternates (C,C), (C,D), (C,C), (C,D), ...

Solution

Preliminary [**1 pt**] Note first that there is a unique NE: (D,D). Moreover the NE achieves for both players the min-max payoff. Hence the smallest discount factor will always be achieved using the punishment of reverting to NE forever.

(a) [**2 pts**] One-step-deviation principle $\Rightarrow$ ROW cannot gain by deviating, so we only have to worry about COL deviating. Payoffs to COL for complying and deviating are

$$\begin{aligned} &\text{Comply} \quad 4 + 4\delta + 4\delta^2 + \ldots &&= \frac{4}{1-\delta} \\ &\text{Deviate} \quad 5 + \delta + \delta^2 + \ldots &&= 5 + \frac{\delta}{1-\delta} \end{aligned}$$

COL (weakly) prefers Comply to Deviate if and only if

$$\delta \geq 1/4$$

(b) [**4 pts**] If play in the current period is supposed to be (C,C) then ROW cannot gain by deviating. Payoffs to COL for complying and deviating are

$$\begin{array}{lll} \text{Comply} & 4 + 0\delta + 4\delta^2 + \ldots & = \dfrac{4}{1-\delta^2} \\ \text{Deviate} & 5 + \delta + \delta^2 + \ldots & = 5 + \dfrac{\delta}{1-\delta} \end{array}$$

Hence COL (weakly) prefers Comply to Deviate if

$$4\delta^2 - \delta - 1 \geq 0$$

which reduces to

$$\delta \geq \frac{1+\sqrt{17}}{8}$$

If play in the current period is supposed to be (D,C) then *both* ROW and COL can gain by deviating. Payoffs to ROW for complying and deviating are

$$\begin{array}{lll} \text{Comply} & 2 + 4\delta + 2\delta^2 + \ldots & = \dfrac{2+4\delta}{1-\delta^2} \\ \text{Deviate} & 4 + \delta + \delta^2 + \ldots & = 4 + \dfrac{\delta}{1-\delta} \end{array}$$

Hence ROW (weakly) prefers Comply to Deviate if

$$3\delta^2 + 3\delta - 2 \geq 0$$

which reduces to

$$\delta \geq \frac{-3+\sqrt{33}}{6}$$

Payoffs to COL for complying and deviating are

$$\begin{array}{lll} \text{Comply} & 0 + 4\delta + 0\delta^2 + \ldots & = \dfrac{4\delta}{1-\delta^2} \\ \text{Deviate} & 1 + \delta + \delta^2 + \ldots & = \dfrac{1}{1-\delta} \end{array}$$

Hence COL (weakly) prefers Comply to Deviate if

$$\delta \geq \frac{1}{3}$$

In order for this to be SGPE we must have

$$\delta \geq \max\left\{\frac{1+\sqrt{17}}{8}, \frac{-3+\sqrt{33}}{6}, \frac{1}{3}\right\} = \frac{1+\sqrt{17}}{8}$$

(c) [**3 pts**] If play in the current period is supposed to be (C,D) then ROW's payoff if he complies is

$$(-2) + 4\delta + (-2)\delta^2 + \ldots$$

The long run average is

$$(1-\delta)[(-2) + 4\delta + (-2)\delta^2 + \ldots] < 1$$

Since ROW's minmax long run average payoff is 1, this means ROW can always deviate and gain for every $\delta > 0$. **Hence there is no δ for which this is a SGPE.**

Differentiated Commodities Two firms produce differentiated commodities for sale in a single market. The firms have 0 fixed costs and constant marginal costs $c_1, c_2 \geq 0$. The market demands are

$$\begin{aligned} q_1 &= (1 - p_1 + 2p_2)^+ \\ q_2 &= (2 + p_1 - p_2)^+ \end{aligned}$$

Suppose first that the firms choose prices simultaneously so that the firms are playing a strategic form game.

(a) For what values of c_1, c_2 (if any) is there a (pure strategy) Nash equilibrium in pure strategies in which both firms sell a positive quantity? For these values (if any), find (at least) one.

(b) For what values of c_1, c_2 (if any) is there a (pure strategy) Nash equilibrium in pure strategies in which *only firm 1* sells a positive quantity? For these values (if any), find (at least) one.

(c) For what values of c_1, c_2 (if any) is there a (pure strategy) Nash equilibrium in pure strategies in which *only firm 2* sells a positive quantity? For these values (if any), find (at least) one.

In all of the above, don't worry about knife-edge cases in which one firm is indifferent to operating or not.

Now suppose that firm 1 chooses its price first and firm 2 observes the choice of firm 1 before choosing its price, so that the firms are playing an sequential/extensive form game.

(d) For what values of c_1, c_2 (if any) is there a (pure strategy) subgame perfect equilibrium in which both firms sell a positive quantity? For these values (if any), find (at least) one.

(e) For what values of c_1, c_2 (if any) is there a (pure strategy) subgame perfect equilibrium in which *only firm 1* sells a positive quantity? For these values (if any), find (at least) one.

(f) For what values of c_1, c_2 is there a (pure strategy) subgame perfect equilibrium in which *only firm 2* sells a positive quantity? For these values (if any), find (at least) one.

In all of the above, don't worry about knife-edge cases in which one firm is indifferent to operating or not.

Solution

The profit functions of the firms are

$$\begin{aligned}\Pi_1 &= (1 - p_1 + 2p_2)(p_1 - c_1) \\ \Pi_2 &= (2 + p_1 - p_2)(p_2 - c_2)\end{aligned}$$

provided profits are positive (ignoring knife-edge cases); otherwise profits are 0.

(a) [**2 pts**] If both firms sell positive quantities then profits are strictly positive and best responses are determined by the first order condition

$$\begin{aligned}0 = \partial\Pi_1\partial p_1 &= 1 + c_1 - 2p_1 + 2p_2 \\ 0 = \partial\Pi_2\partial p_2 &= 2 + c_2 + p1 - 2p_2\end{aligned}$$

This gives best responses as follows

$$\begin{aligned}p_1 &= (1/2)(1 + c_1 + 2p_2) \\ p_2 &= (1/2)(2 + c_2 + p_1)\end{aligned}$$

Solving simultaneously gives

$$\begin{aligned}p_1^* &= 2 + c_1 + c_2 \\ p_2^* &= (1/2)(5 + c_1 + 2c_2)\end{aligned}$$

Check that at prices p_1^*, p_2^* both firms are making positive profits. Hence for all c_1, c_2 this is the unique NE in which both firms sell positive quantities.

(b) [**2 pts**] If only firm 1 sells a positive quantity then equilibrium prices p_1^*, p_2^* must have the property that the demand for firm 2's product is 0; ignoring knife-edge cases this means

$$2 + p_1^* - p_2^* < 0$$

If this is true then (by the best response calculated above) we must have

$$p_1^* = (1/2)(1 + c_1 + 2p_2) > (1/2)(1 + c_1 + 4 + p_1^*)$$

which is absurd. Hence there cannot be an equilibrium in which only firm 1 sells a positive quantity.

(c) [**2 pts**] If only firm 2 sells a positive quantity then equilibrium prices p_1^*, p_2^* must have the property that the demand for firm 1's product is 0; ignoring knife-edge cases this means

$$1 - p_1 + 2p_2 < 0$$

and then (by the best response calculated above)

$$p_2^* = (1/2)(2 + c_2 + p_1) > (1/2)(2 + c_2 + 1 + 2p_2^*)$$

which is absurd. Hence there cannot be an equilibrium in which only firm 2 sells a positive quantity.

(d) [**2 pts**] If firm 1 chooses its price p_1 first and both firms sell positive quantities then firm 2's best response is as above so firm 1 maximizes

$$(1 - p_1 + 2 + c_2 + p_1)(p_1 - c_1) = (3 + c_2)(p_1 - c_1)$$

But this has no maximum, so there is no such SGPE. [Note: non-existence does not violate any general theorems because profits can be unbounded!]

(e) [**1 pt**] Suppose firm 1 sets a very high price. Then firm 2 can always set a price that is above c_2 and for which it sells a positive quantity. Hence the best response for firm 2 involves selling a positive quantity so we are back in (d). Hence there is no such equilibrium.

(f) [**1 pt**] As in (d) there is no such equilibrium.