Spring 2015

1. **Economy with Quasi-Linear Preferences**

Consider n consumers with the following quasi-linear preferences: consumer i's utility from consuming $(x_i, m_i) \in \mathbb{R}_+^{L-1} \times \mathbb{R}$ is given by $v_i(x_i) + m_i$ (note that m_i can be any real number). Assume that v_i is continuously differentiable, concave, strictly increasing in $\mathbb{R}_+^{L-1}$ and $\lim_{x_{i,\ell} \to 0} \frac{\partial v_i(x_i)}{\partial x_{i,\ell}} = \infty$ given any $x_{i,-\ell} \in \mathbb{R}_+^{L-2}$. An allocation $(\mathbf{x}, \mathbf{m}) \in \mathbb{R}_+^{(L-1)n} \times \mathbb{R}^n$ in this economy is feasible if $\sum_{i=1}^{n} x_i \leq r$ and $\sum_{i=1}^{n} m_i = M$, where $r \in \mathbb{R}_{++}^{L-1}$ and $M > 0$ are the total resources that are available in this economy. Answer the following questions.

(a) A feasible allocation $(\mathbf{x}, \mathbf{m})$ is Pareto efficient if and only if $\mathbf{x}$ solves the following problem:

$$(P) \quad \max_{\mathbf{x} \in \mathbb{R}_+^{(L-1)n}} \sum_{i=1}^{n} v_i(x_i) \ \text{ s.t. } \ \sum_{i=1}^{n} x_i \leq r.$$

Prove this statement in two steps.

(i) If $\mathbf{x}$ solves (P), then any feasible allocation $(\mathbf{x}, \mathbf{m})$ is Pareto efficient. Explain why this is the case briefly.

(ii) Prove the other direction. Show that, if a feasible allocation $(\mathbf{x}, \mathbf{m})$ is Pareto efficient, then $\mathbf{x}$ must solve (P).

(b) Write down the necessary and sufficient condition (Kuhn-Tucker condition) for the optimal solution for the problem (P). Explain why it is necessary and sufficient briefly.

(c) Let $\mathbf{x}^*$ be a solution for (P). Show that there exists $p^* \in \mathbb{R}_{++}^{L}$ such that $(\mathbf{x}^*, \mathbf{m}^*, p^*)$ is a Walrasian equilibrium with transfer for any $\mathbf{m}^*$ such that $\sum_{i=1}^{n} m_i^* = M$ (Do not just appeal to the second welfare theorem).

Answer for Q1

(a) (3 pts.)
(a-i)

If $\mathbf{x}$ solves (P), then any feasible $(\mathbf{x}, \mathbf{m})$ maximizes $\sum_{i=1}^{n} (v_i(x_i) + m_i)$ within the set of feasible allocations. Then clearly any such $(\mathbf{x}, \mathbf{m})$ must be Pareto-efficient.

(a-ii)

Suppose that a feasible allocation $(\mathbf{x}, \mathbf{m})$ is Pareto efficient, but $\mathbf{x}$ does not solve (P), i.e. there exists $\mathbf{x}' \in \mathbb{R}_+^{(L-1)n}$ such that $\sum_{i=1}^{n} x_i' \leq r$ and $\sum_{i=1}^{n} v_i(x_i') > \sum_{i=1}^{n} v_i(x_i)$. Then we can find transfer $\mathbf{m}'$ such that $\sum_{i=1}^{n} m' = M$ and $v_i(x_i') + m_i' > v_i(x_i) + m_i$ for every i. This is a contradiction.

(b) (3 pts.) The Kuhn-Tucker condition for (P) is

$$\nabla v_i(x_i) - q = 0 \text{ for } i = 1, ..., n \text{ for some } q \in \mathbb{R}_{++}^{L-1}$$

(Note: (i) the solution must be an interior point $\mathbf{x} \gg \mathbf{0}$, (ii) $\nabla v_i(x_i)$ cannot have any 0 element because it is continuously differentiable, strictly increasing and concave). This is necessary because the constraint qualification is satisfied ((i) the derivative of the binding constraint $\sum_{i=1}^{n} x_i = r$ is nondegenerate or (ii) the constraint is linear and $r \gg 0$, i.e. $\exists$ an interior point in the constraint set). It is sufficient because v_i is concave and the constraint is linear.

(c) (4 pts.) Let $p^* = (q, 1)$. Take any $\mathbf{m}^*$ such that $\sum_{i=1}^{n} m_i^* = M$. Assign wealth $w_i = qx_i^* + m_i^*$ to consumer i. Then (i) $\sum_{i=1}^{n} w_i = q \cdot r + M$ (i.e. the sum of wealth is the total wealth with respect to p^*) and (ii) (x_i^*, m_i^*) is the optimal consumption for consumer i given wealth w_i and price p^*, because (x_i^*, m_i^*) satisfies $\nabla v_i(x_i^*) - q = 0$ and $w_i = qx_i^* + m_i^*$, which are necessary and sufficient condition for the optimal consumption given wealth w_i and price p^*. Hence $(\mathbf{x}^*, \mathbf{m}^*, p^*)$ is a Walrasian equilibrium with transfer.

2. **Insurance**

Kenny is considering to purchase a car insurance. There are three possibilities: he may have no car accident, or a minor accident, or a major car accident with equal probability (Kenny is not a good driver). If he is involved with a minor accident, he would lose \$1000. If he is involved with a major accident, he would lose \$5000. An insurance is given by $(\delta, B_{\min}, B_{maj})$ where δ is the premium to pay in advance, $B_{\min}$ and B_{maj} are the benefits Kenny would receive in the case of a minor accident and a major accident respectively. He is an expected utility maximizer with a strictly increasing vNM utility function $u(\cdot)$. So his expected utility from an insurance $(\delta, B_{\min}, B_{maj})$ is $\frac{1}{3}u(-\delta)+\frac{1}{3}u(B_{\min}-\delta-1000)+\frac{1}{3}u(B_{maj}-\delta-5000)$. Answer the following questions.

(a) Consider an insurance $(\delta, B_{\min}, B_{maj}) = (200, 1000, 3000)$. Does Kenny prefer this insurance to no insurance (for any strictly increasing u)? If so, prove it. If you think that it depends on the shape of u, then find u such that Kenny prefers not to buy this insurance.

(b) Consider an insurance $(\delta, B_{\min}, B_{maj}) = (200, 1200, 1200)$. Does Kenny prefer this insurance to no insurance? If so, prove it. If you think that it depends on the shape of u, then find u such that Kenny prefers not to buy this insurance.

(c) Consider another insurance $(\delta, B_{\min}, B_{maj}) = (200, 400, 800)$. Suppose that Kenny is risk averse: u is strictly concave. Does Kenny prefer this insurance to no insurance? If so, prove it. If you think that it depends on the shape of u, then find u such that Kenny prefers not to buy this insurance.

(d) Consider the following two insurances that are fair (i.e. $-\delta+\frac{1}{3}B_{\min}+\frac{1}{3}B_{maj}=0$): $\left(\delta', B'_{\min}, B'_{maj}\right) = (100, 100, 200)$ and $\left(\delta'', B''_{\min}, B''_{maj}\right) = (100, 0, 300)$. Suppose that these insurances are divisible. If Kenny purchases $x \geqq 0$ units of the first insurance and $y \geqq 0$ units of the second, then his total insurance can be represented as $\left(x\delta'+y\delta'', xB'_{\min}+yB''_{\min}, xB'_{maj}+yB''_{maj}\right)$. Suppose that u is strictly concave. Discuss how Kenny would combine these two insurances optimally.

Answer for Q2

(a) (2.5 pts.) This insurance generates the following outcome: $(-200, -200, -2200)$ (each equally likely). Without the insurance, Kenny would have the following $(0, -1000, -5000)$. Since the cumulative distribution function for the former does not first order stochastically dominate the CDF of the latter, there is some strictly increasing u such that Kenny would prefer not to buy this insurance. For example, suppose that $u(x) = \max\{0.001x, 10x + 99.99\}$. Then Kenny would not purchase this insurance.

(b) (2.5 pts.) This one generates $(-200, 0, -4000)$. The CDF of this first order stochastically dominate the CDF of no insurance. Hence Kenny prefers to buy this insurance as long as u is strictly increasing.

(c) (2.5 pts.) Consider another insurance $(200, 200, 400)$. This means that, pay 200 if nothing happens and receive 200 if a major accident occurs. It is easy to check that (1) no insurance is a mean-preserving spread of this or (2) the integral of CDF for this is always smaller than the integral of CDF for no insurance (condition for second order stochastic dominance). Hence every risk averse individual would prefer to buy this insurance. Since the insurance in the question is clearly better than this, every risk averse individual would prefer to buy it as well.

(d) (2.5 pts.) Suppose that Kenny can pick any $(\delta, B_{\min}, B_{maj})$ that is fair. Then the optimal choice for Kenny would be $\left(\delta^*, B^*_{\min}, B^*_{maj}\right) = (2000, 1000, 5000)$, which is full insurance. Kenny can combine the above two insurances to generate this full insurance by choosing $x = 10$ and $y = 10$. Hence the optimal choice for Kenny is $(x^*, y^*) = (10, 10)$.

#3) (Subgame Perfect Bargaining) Two players must divide \$1 according to the following procedure: Player 1 proposes a division $(x, 1-x)$ (with $0 \leq x \leq 1$); Player 2 can Accept or Reject. If Player 2 Accepts the proposed division is implemented, otherwise both agents get 0.

Players 1 and 2 care about their own consumption and also about fairness; if the outcome is (x_1, x_2) their utilities are

$$\begin{aligned} u_1(x_1, x_2) &= x_1 - \theta_1 |x_1 - x_2| \\ u_2(x_1, x_2) &= x_2 - \theta_2 |x_1 - x_2| \end{aligned}$$

where $\theta_1, \theta_2 \geq 0$ are parameters that measures how much players care about fairness.

Find all the pure strategy subgame perfect equilibria of this game. Of course your answer will depend on the parameters θ_1, θ_2.

You may find it helpful to first graph players' utility for divisions $(x, 1-x)$ as a function of x and think about how the parametes θ_1, θ_2 affect the graph.

4) **(Repeated Games)** For the two stage games G below, consider the infinitely repeated game in which players use the discount factor $\delta \in (0, 1)$.

G

	L	R
U	3.1,1	1,3.1
D	0,2	2,0

(a) Which long term average payoffs can be supported as subgame perfect equilibria (in pure strategies) for δ very close to 1? (Provide an explicit description of these payoffs.)

(b) Which long term average payoffs can be supported as subgame perfect equilibria (in pure strategies) for $\delta = .9$? Give a complete argument.

Solution (sketch) to #3) First determine behavior of Player 2. If $(x, 1-x)$ is offered then Player 2 will Reject if $u_2(x, 1-x) < 0$, Accept if $u_2(x, 1-x) > 0$. If $x \geq .5$, solving shows Player 2 Accepts if $x < (1+\theta_2)/(1+2\theta_2)$, Rejects if $x > (1+\theta_2)/(1+2\theta_2)$. In order that the optimization problem of Player 1 have a solution, it must be that Player 2 Accepts when $x = (1+\theta_2)/(1+2\theta_2)$. If $x < .5$ Player 2 Accepts if $(1-\theta_2) - (1-2\theta_2)x > 0$, Rejects if $(1-\theta_2) - (1-2\theta_2)x < 0$ and can do anything if $(1-\theta_2) - (1-2\theta_2)x = 0$ (because Player 1 will never choose $x < .5$.

If $\theta_1 < .5$ the Player 1 will choose $x = (1+\theta_2)/(1+2\theta_2)$: this is the biggest the Player 2 will Accept. If $\theta_1 \geq .5$ Player 1 will choose $x = .5$ independently of what Player 2 plans.

Solution (sketch) to #4) The pure minmax vector is $(2, 2)$. By the Folk Theorem, any feasible adn strictly individually rational payoff can be attained in SGPE if players are δ is close enough to 1; this set is

$$X = (x_1, x_2) : x_1 + x_2 = 4.1, x_1 > 2, x_2 > 2\}$$

If $\delta = .9$ then **the infinitely repeated game has no SGPE**. To see this note that any SGPE outcome must lie in the set X above. But then at some point on the equilibrium path play must at some point be either UL or UR; say UL – the other case is similar. Subgame perfection guarantees that the equilibrium continuation play is in X so Player 2's continuation payoff in equilibrium play is at most 2.1 (long run average). If Player 2 deviates to R in the current period; this will give 3.1 in the current period and at least the minmax payoff of 2 in every succeeding payoff so the long run average will be $(3.1)(1 - \delta) + (2)(\delta) = 2.31 > 2.1$. Hence Player 2 can gain by deviating – so there is no SGPE.