Spring 2014

1. **Classical Equilibrium Existence Theorem**

Consider a pure exchange economy $\mathcal{E}^{pure} = (\{X_i, \succeq_i, e_i\}_{i \in I})$ with free disposal technology, where $\mathbb{X}_i = \mathbb{R}^L_+$ and $\succeq_i$ is rational, continuous and strictly convex. Also assume that i's upper contour set $U_i(x_i') = \{x_i \in X_i | x_i \succeq_i x_i'\}$ is bounded for every $x_i' \in X_i$ for any i (hence $\succeq_i$ is satiated, i.e. there exists $\widehat{x}_i \in X_i$ such that $\widehat{x}_i \succeq_i x_i$ for all $x_i \in X_i$). Answer the following questions.

(a) Define Walrasian equilibrium in this economy.

(b) Prove that Walrasian demand correspondence $x_i(p)$ can defined on $\mathbb{R}^L_+/\{\mathbf{0}\}$ and it is a continuous function.

(c) Prove that there exists a Walrasian equilibrium in this economy. Do not use any Walrasian equilibrium existence theorem (but you can use a fixed point theorem).

Answer for Q1

(a) (2 pts.) $(x^*, p^*) \in X \times \mathbb{R}^L_+$ is a Walrasian equilibrium if (1) x_i^* maximizes consumer i's utility given the budget constraint $p^* \cdot x_i \leq p^* \cdot e_i$ for every i and (2) it satisfies $\sum_i x_i^* \leq \sum_i e_i$.

(b) (4 pts) First we are going to apply the Maximum theorem to the consumer problem. Note that the budget set is lower hemicontinuous, but not upper hemicontinuous with respect to p when some prices are 0. Given any $p \in \mathbb{R}^L_+/\{\mathbf{0}\}$, pick $x_i' \in X_i$ such that $p \cdot x_i' < p \cdot e_i$. Such x_i' exists because $e_i \gg 0$ (hence $p \cdot e_i > 0$). Consider the following modified consumer problem:

$$\max_{x_i \in X_i} u_i(x_i) \text{ s.t. } p^* \cdot x_i \leq p^* \cdot e_i \text{ and } x_i \succeq_i x_i'.$$

Clearly the added condition $x_i \succeq_i x_i'$ does not change the optimal solution of the problem locally around this price p. Thus the solution of this problem is still $x_i(p)$ locally.

Now it can be shown that the constraint set for this problem is continuous:

- Upper hemicontinuity: uhc follows from the assumption that the upper contour set is bounded (hence compact by the continuity of the preference) and independent of the parameter p.

- Lower hemicontinuity: lhc still holds because x_i' is chosen as an interior point of the budget set.

Since the constraint set is nonempty and continuous around p and the objective function is continuous with respect to x_i and p (trivially), $x_i(p)$ is an (non empty) upper hemicontinuous correspondence locally. This holds for every $p \in \mathbb{R}^L_+/\{\mathbf{0}\}$. Thus $x_i(p)$ is an upper hemicontinuous correspondence on $\mathbb{R}^L_+/\{\mathbf{0}\}$.

Finally, $x_i(p)$ is a function because $\succeq_i$ is strictly convex. Hence $x_i(p)$ is a continuous function in $\mathbb{R}^L_+/\{\mathbf{0}\}$.

(c) (4 pts.) Let $z_\ell(p) = \sum_i x_{i,\ell}(p) - \sum_i e_{i,\ell}$ be the excess demand function for good ℓ and Δ be the price simplex. $z_\ell(p)$ is well defined on Δ and continuous because $x_i(p)$ is nonempty and continuous on $p \in \mathbb{R}^L_+/\{\mathbf{0}\}$. The proof goes as follows (for example).

- Define $f : \Delta \to \Delta$ by $f_\ell(p) := \frac{p_\ell + max\{0, z_\ell(p)\}}{1 + \sum_\ell \max\{0, z_\ell(p)\}}, \ell = 1, ..., L$.

- f is continuous. Hence there exists a fixed point $p^* \in \Delta$ of f by Brouwer's fixed point theorem.

- Then $z_\ell(p^*) \leq 0$ holds for $\ell = 1, ..., L$. This can be shown as follows.

- Multiply both sides by $z_\ell(p^*)$ and sum them up across all goods. Then we have

$$p^* \cdot z(p^*) = \frac{p^* \cdot z(p^*) + \sum_{\ell \in L} z_\ell(p^*) \max\{0, z_\ell(p^*)\}}{1 + \sum_{\ell \in L} \max\{0, z_\ell(p^*)\}}.$$

- Note that $p^* \cdot z(p^*) \leq 0$ (Walras' law does not hold because of satiated preferences). Hence

$$\begin{aligned} \sum_{\ell \in L} z_\ell(p^*) \max\{0, z_\ell(p^*)\} &= p^* \cdot z(p^*) \sum_{\ell \in L} \max\{0, z_\ell(p^*)\} \\ &\leq 0. \end{aligned}$$

  This implies that $z_\ell(p^*) \leq 0$ for every ℓ.

Since $x(p^*)$ satisfies (1) and (2) in (a), $(x(p^*), p^*)$ is a Walrasian equilibrium.

2. **Subjective Expected Utility**

Let S be a set of finite states, X be a space of outcome, and Π be the space of simple lotteries on X (i.e. the set of finite support distributions on X). Let $f : S \to \Pi$ be an act and $\mathcal{F}$ be the set of all acts. Consider a decision maker who has a preference $\succeq$ on $\mathcal{F}$. Suppose that $\succeq$ can be represented by a function $U : \mathcal{F} \to \mathbb{R}$ ($f \succeq g$ if and only if $U(f) \geq U(g)$) that satisfies $U(\alpha f + (1-a)g) = \alpha U(f) + (1-\alpha)g$ for any $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$. Answer the following questions.

(a) When can you find such U to represent $\succeq$? Describe three axioms on $\succeq$ to guarantee the existence of such U (no proof is needed).

(b) Prove that there exists $u_s : X \to \mathbb{R}, s \in S$ such that for any $f, g \in \mathcal{F}$, the following holds:

$$f \succeq g \text{ if and only if } \sum_{s \in S} \left[\sum_{x \in \mathrm{supp}(f_s)} u_s(x) f_s(x) \right] \geq \sum_{s \in S} \left[\sum_{x \in \mathrm{supp}(g_s)} u_s(x) g_s(x) \right].$$

(c) Suppose that $\succeq$ satisfies the following axiom:

$$(*) \text{ For any } p, q \in \Pi, \text{ if } (p, ..., p) \succeq (q, ..., q), \text{ then } (p, f_{-s}) \succeq (q, f_{-s}) \text{ for any } f \in \mathcal{F} \text{ and } s \in S,$$

where (p, f_{-s}) is the act that is obtained by replacing f_s with lottery $p \in \Pi$.

- Prove that there exists a distribution μ on S and $u : X \to \mathbb{R}$ such that for any $f, g \in \mathcal{F}$, the following holds:

$$f \succeq g \text{ if and only if } \sum_{s \in S} \mu(s) \left[\sum_{x \in \mathrm{supp}(f_s)} u(x) f_s(x) \right] \geq \sum_{s \in S} \mu(s) \left[\sum_{x \in \mathrm{supp}(g_s)} u(x) g_s(x) \right].$$

Answer for Q2

(a)(2 pts.)
For example, the following three axioms do the job.

- **(Rationality):** $\succeq$ on $\mathcal{F}$ is complete and transitive.
- **(Archimedian/Continuity Axiom):** For any $f, g, h \in \mathcal{F}$ such that $f \succ g \succ h$, there exist $a, b \in (0,1)$ such that

$$af + (1-a)\,h \succ g \succ bf + (1-b)h.$$

- **(Independence):** For any $f, g, h \in \mathcal{F}$ and $a \in (0,1)$,

$$f \succ g \Rightarrow af + (1-a)h \succ ag + (1-a)h.$$

(b)(4 pts)

- Fix any $f^0 \in \mathcal{F}$ and normalize $U(f^0)$ to 0. Note that, for any $f \in \mathcal{F}$, $\frac{1}{S}f + \frac{S-1}{S}f^0 = \frac{1}{S}\sum_{s\in S}(f_s, f^0_{-s})$ holds.
- Then, by the linearity of U, $\frac{1}{S}U(f) + \frac{S-1}{S}U\left(f^0\right) = \frac{1}{S}\sum_{s\in S}U(f_s, f^0_{-s})$. Hence $U(f) = \sum_{s\in S}U_s(f_s)$, where U_s is defined as $U_s(f_s) := U(f_s, f^0_{-s})$.
- Linearity of U implies linearity of each U_s (take a convex combination of two acts that differ only in one state). That is, $U_s\,(ap + (1-a)q) = aU_s\,(p) + (1-a)U_s\,(q)$ for any $p, q \in \Pi$ and $a \in [0,1]$.[1]
- This means that U_s can take the expected utility form, i.e. there exists u_s such that $U_s(f_s) = \sum_{x\in \mathrm{supp}(f_s)} u_s(x)f_s(x)$ by the standard argument from the expected utility theorem (formally, this can be proved by an induction argument with respect to the size of the support of simple distributions by defining $u_s(x) := U_s(\delta_x)$ for each $x \in X$, where δ_x is the Dirac measure on x).

(c)(4 pts.)

- By the result in (b), there exists $u_s, s \in S$ such that $f \succeq g$ if and only if $\sum_{s\in S}\left[\sum_{x\in \mathrm{supp}(f_s)} u_s(x)f_s(x)\right] \geq \sum_{s\in S}\left[\sum_{x\in \mathrm{supp}(g_s)} u_s(x)g_s(x)\right]$.

[1] For any $p, q \in \Pi, a \in [0,1]$ and f,

$$U\left(a\left(p, f_{-s}\right) + (1-a)\left(q, f_{-s}\right)\right) = aU\left(p, f_{-s}\right) + (1-a)U\left(q, f_{-s}\right).$$

This implies

$$U_s\left(ap + (1-a)q\right) = aU_s\left(p\right) + (1-a)\,U_s\left(q\right).$$

- $(*)$ implies that $(p, f_{-s}) \succeq (q, f_{-s}) \Leftrightarrow (p, f_{-s'}) \succeq (q, f_{-s'})$ for any $p, q \in \Pi$, $s, s' \in S$ and $f \in \mathcal{F}$. Hence $\sum_{x \in \mathrm{supp}(p)} u_s(x)p(x) \geq \sum_{x \in \mathrm{supp}(q)} u_s(x)q(x)$ if and only if $\sum_{x \in \mathrm{supp}(p)} u_{s'}(x)p(x) \geq \sum_{x \in \mathrm{supp}(q)} u_{s'}(x)q(x)$ for any $p, q \in \Pi$, $s, s' \in S$ This means that $\sum_{x \in \mathrm{supp}(p)} u_s(x)p(x)$ and $\sum_{x \in \mathrm{supp}(p)} u_{s'}(x)p(x)$ represent the same preference on Π.

- Pick any $\widehat{s}$ and define $u(x) := u_{\widehat{s}}(x)$. Then, by the uniqueness of the expected utility representation on Π, there exist $A_s > 0$ and B_s such that $u_s = A_s u + B_s$ for every $s \in S$.[2] Hence

$$\sum_{s \in S} \left[\sum_{x \in \mathrm{supp}(f_s)} u_s(x) f_s(x) \right] = \sum_{s \in S} A_s \left[\sum_{x \in \mathrm{supp}(f_s)} u(x) f_s(x) \right] + \sum_{s \in S} B_s.$$

- Subtract $\sum_{s \in S} B_s$, divide by $\sum_s A_s$ and define $\mu(s') := \frac{A_{s'}}{\sum_s A_s}$. This affine transformation does not affect the preference. Then we have the desired representation $\sum_{s \in S} \mu(s) \left[\sum_{x \in \mathrm{supp}(f_s)} u(x) f_s(x) \right]$.

[2] If the preference is trivial (complete indifference), then such (A_s, B_s) is not unique. But it is still possible to pick some (A_s, B_s) such that $A_s > 0$.

Qualifying Exam Questions - Zame

Solutions

#3) **(Perfect Bayesian Equilibrium)** Consider an environment with one Firm and one Consumer. The Firm produces a divisible good at 0 fixed cost and constant marginal cost $k = 1$. If the Consumer purchases x units of the good at a per-unit price of p then the firm makes profit

$$(p-1)x$$

and the Consumer experiences utility that depends on the state of nature: Good G or Bad B:

$$\begin{aligned} u_G &= 6x - x^2 - px \\ u_B(x) &= 2x - x^2 - px \end{aligned}$$

(In what follows, assume the money endowment of the Consumer is so large that the non-negativity constraint never binds.)

The true weather is Good or Bad with equal probability; this is common knowledge. The Firm learns the true weather; after learning the true weather the Firm offers the good at a price p. The Consumer does not learn the true weather but observes the offered price and buys as much or as little of the good as desired. The Firm seeks to maximize profit; the Consumer seeks to maximize (expected) utility.

This defines a Bayesian game between the Firm and the Consumer: the Firm observes the offers a price, the Consumer chooses a quantity. We are interested in pure strategy equilibria only.

(a) Find the pooling Perfect Bayesian Nash equilibrium (i.e. a PBNE in which the firm offers the same price in each state) that is best for the firm in the sense of yielding the firm the largest *ex ante* expected profit (the largest expected profit *before* the Firm learns the weather).

(b) Find a separating Perfect Bayesian Nash equilibrium (i.e. a PBNE in which the firm offers different prices in each state) in which the firm makes positive profits in both states. Is this separating equilibrium unique?

(c) Show that the pooling PBNE equilibrium you found in (a) is better for the firm than *every* separating PBNE equilibrium .

Grading Suggestions

(a) (4 pts) Derive optimal choice of consumer (1pt) and optimal profit of the firm (1 pt); show this can be supported in PBNE (2pts)

(b) (4pts) Find BNE (2 pts); show non-uniqueness of PBNE (2 pts)

(c) (2 pts) Just compare unique separating with best pooling. If student did not find unique separating just compare all separating with best pooling.

Solution (a) We look for pooling PBNE. Suppose the firm's strategy is to offer the price p in both states. The Consumer maximizes expected welfare so chooses x to maximize

$$(1/2)(6x - x^2 - px) + (1/2)(2x - x^2 - px) = 4x - x^2 - px$$

The FOC is

$$4 - 2x - p = 0$$

so the optimal choice of the Consumer is $x = (1/2)(4 - p)$ and the profit of the firm is

$$\Pi(p) = (1/2)(4 - p)(p - 1)$$

The FOC is

$$(1/2)\left[-(p - 1) + (4 - p)\right] = 0$$

so the optimal profit is attained when $p^* = 5/2$ and optimal profit $\Pi^* = 9/8$. (Notice that the Consumer makes positive utility when the state is Good and negative utility when the state is Bad but makes positive utility in expectation.)

Claim: this can be supported as a PBNE. To see this we must specify the strategy of the Consumer:

- If the firm offers $p = 5/2$ the consumer buys $x^* = (1/2)(3/2) = 3/4$ units.
- If the firm offers $p \neq 5/2$, the consumer believes the state is Bad and optimizes accordingly.

What has to be checked is that when the firm offers $p \neq 5/2$ the maximal profit is less than $9/8$ (so the firm does not want to deviate). Given the strategy of the Consumer, the best the Firm can do is to maximize profit given that the Consumer acts as if the state were Bad; i.e. the Consumer maximizes $2x - x^2 - px$. The FOC is

$$2 - 2x - p = 0$$

so $x = (1/2)(2-p)$ and firm profit is $(1/2)(2-p)(p-1)$; this is maximized when $p = 3/2$ and profit is $(1/2)(1/2)(1/2) = 1/8 < 9/8$. Hence the Firm does not want to deviate and this is a PBNE.

(b) We look for separating BNE and then worry about perfection. Suppose the Firm offers the price q when the state is Good and the price $p \neq q$ when the state is Bad. The Consumer does not know the state but infers it from the action of the Firm. Because the Firm is optimizing it must be that the Firm makes the *same* profit in both states (otherwise it would always offer whichever price yielded higher profit). Hence a separating BNE must satisfy the profit identity

$$(p-1)(6-p)/2 = (q-1)(2-q)/2$$

In order that profit be positive in the Bad states, we must have $1 < q < 2$; take any such q and solve for p. For example if $q = 3/2$ then profit in the Bad state is $1/8$ and we can solve for p (price in the Good state). [This is not required; it is just algebra.] This gives all the separating BNE.

However not all the separating BNE are perfect: the argument above does not take into account what happens if the firm offers a price $r \neq p, q$. Perfection requires that consumers optimize with respect to some beliefs. The worst belief consumers can have (from the point of view of the firm) is that the state is Bad with probability 1, in which case the firm's profit will be $(r-1)(2-r)/2$; if $r = 3/2$ then profit will be $1/8$. At a PBNE, this deviation cannot be profitable so at a PBNE with these beliefs we must have

$q = 3/2$. However, consumers could also believe the state is bad with probability $1-\varepsilon$ ini which case the firm's profit will be slightly bigger, so q could also be bigger than 3/2. Hence this PBNE is not unique.

(c) We have shown in part (a) above that the largest profit the firm can make when the state is Bad is 1/8 so in any BNE (even if not perfect) the largest profit the firm can make can be no bigger than 1/8. This is worse than in the pooling PBNE we just found.

4) **(Repeated Games)** The stage game G below is played infinitely often; players use the discount factor $\delta \in (0,1)$. (We have called the infinitely repeated game $G^{\infty}(\delta)$.)

	L	R
U	3,0	-1,-1
D	2,2	0,3

(a) Find a discount factor $\delta \in (0,1)$ and a subgame perfect equilibrium strategy profile σ for $G^{\infty}(\delta)$ in which, on the equilibrium path (i.e. when no deviations have occurred), (D, L) is played in every period.

(b) Find a discount factor $\delta \in (0,1)$ and a subgame perfect equilibrium strategy profile σ for $G^{\infty}(\delta)$ in which, on the equilibrium path (i.e. when no deviations have occurred), (D, L) is played in even periods and (U, R) is played in odd periods on the equilibrium path. (The initial period is period 0 so (D, L) should be played, etc.)

Note: in both parts, you are asked to *find* a discount factor and a strategy profile, not just appeal to an existence theorem.

Grading Suggestions

(a) (5 pts) Strategy: play (D,L), use player-specific NE to punish defector (3pts); find δ (2 pts)

(b) (5pts) Strategy: alternate, use player-specific NE to punish defector (3pts); find δ(2 pts)

Solution (a) Define the strategy profile σ as follows

- $\sigma(\emptyset) = (D, L)$
- for every history $h \neq \emptyset$
  - if h has always been (D, L) or (U, R) then $\sigma(h) = (D, L)$
  - if h has not always been (D, L) or (U, R) and the *first* deviation was (U, L) then $\sigma(h) = (D, R)$
  - if h has not always been (D, L) or (U, R) and the *first* deviation was (D, R) then $\sigma(h) = (U, L))$

In words: play (D, L) unless *one* player has deviated. If ROW deviates play (D, R) forever; if COL deviates, play (U, L) forever. Ignore simultaneous deviations by both players.

To see that σ is a SGPE strategy for some δ use one-step deviation principle. If ROW deviates she gains 1 now and loses 2 every period in the future. Hence deviation is profitable if and only if

$$1 > \delta \left(\frac{2}{1 - \delta} \right)$$

which is the same as saying that deviation is profitable if and only if

$$2/3 > \delta$$

The game is symmetric so the same is true for COL.

In other words: σ is a SGPE strategy profile if $\delta \geq 2/3$.

(b) Define the strategy profile σ as follows

- $\sigma(\emptyset) = (D, L)$
- $\sigma(D, L) = (U, R)$
- for every history $h \neq \emptyset$
  - if h has alternated between (D, L) and (U, R) and the length of h is even then $\sigma(h) = (D, L)$

– if h has alternated between (D, L) and (U, R) and the length of h is odd then $\sigma(h) = (U, R)$

– if h has not alternated between (D, L) or (U, R) and the *first* deviation was by ROW then $\sigma(h) = (D, R)$

– if h has not alternated between (D, L) or (U, R) and the *first* deviation was by COL then $\sigma(h) = (U, L)$

– ignore any simultaneous deviations by both players

In words: alternate $(D, L), (U, R)$ unless *one* player has deviated. If ROW deviates play (D, R) forever; if COL deviates, play (U, L) forever.

To see that σ is a SGPE strategy for some δ use one-step deviation principle. If ROW deviates she gains 1 now and loses *at least* the value of alternating between -1 and +2 every period in the future. Hence if deviation is profitable then

$$1 > \left(\frac{-1}{1-\delta^2}\right) + \delta\left(\frac{\delta}{1-\delta^2}\right)$$

Simplying shows that this impies

$$\delta^2 + 2\delta - 2 < 0$$

Hence σ is a SGPE strategy profile if the discount factor δ is bigger than the biggest solution to the equation

$$\delta^2 + 2\delta - 2 = 0$$

It is sufficient that $\delta > .8$.