Spring 2013

1. Pareto Efficient Allocation and Social Welfare Maximization

Consider a private ownership economy $\mathcal{E}^{priv} = \left(\left\{ \mathbb{R}^L_+, \succeq_i, e_i \right\}_{i=1,...,I}, \{Y_j\}_{j=1,...,J}, \{\theta_{i,j}\}_{i,j} \right)$ where $\succeq_i$ can be represented by a concave utility function u_i for each i and the total production set $Y = \sum_{j=1}^{J} Y_j$ is convex. Answer the following questions.

(a) Let U be the **utility possibility set** given by

$$U = \left\{ u \in \mathbb{R}^I | \exists (x, y) \in A, u \leq u(x) \right\},$$

where A is the set of feasible allocations and $u(x) = (u_1(x_1), ..., u_I(x_I))^\top$. Show that U is a convex set.

(b) Define **Pareto efficient allocation** in this economy. Show that, for any Pareto efficient allocation (x^*, y^*), $u(x^*)$ is a boundary point of U.

(c) Let $(x^*, y^*) \in \mathbb{R}^{L \times I}_+ \times \prod_{j=1}^{J} Y_j$ be any Pareto efficient allocation. Show that it is a solution for the following optimization problem for some nonnegative weights $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_I) \left(\neq \mathbf{0} \in \mathbb{R}^L \right)$.

$$\max_{(x,y) \in A} \sum_{i=1}^{I} \lambda_i u_i(x_i)$$

Answer for Q1:

(a) (30%) Take any $u, u' \in U$. Then there exists $(x, y), (x', y') \in A$ such that $u \leq u(x)$ and $u' \leq u(x')$. Then for any $\alpha \in [0, 1]$,

$$\begin{aligned} \alpha u + (1-\alpha) u' &\leq \alpha u(x) + (1-\alpha) u\left(x'\right) \\ &\leq u\left(\alpha x + (1-\alpha) x'\right) \text{ (by concavity).} \end{aligned}$$

What is left to show is that $\alpha x + (1-\alpha) x'$ is feasible given some production. Clearly $\sum_{i=1}^{I} (\alpha x_i + (1-\alpha) x_i') = \alpha \sum_{j=1}^{J} y_j + (1-\alpha) \sum_{j=1}^{J} y_j'$ because $(x, y), (x', y') \in A$. Since the total production set is convex, there exists $y'' \in \prod_{j=1}^{J} Y_j$ such that $\sum_{j=1}^{J} y_j'' = \alpha \sum_{j=1}^{J} y_j + (1-\alpha) \sum_{j=1}^{J} y_j'$. Therefore $(\alpha x + (1-\alpha) x', y'') \in A$.

(b) (30%) The definition of Pareto-efficient allocation is a usual one. Take any Pareto-efficient allocation $(x^*, y^*) \in A$. Then clearly $u(x^*) \in U$. Hence, if $u(x^*)$ is not a boundary point of U, then it must be an interior point of U. This means that there exists u' and $(x', y') \in A$ such that $u(x^*) \ll u' \leq u(x')$, which is a contradiction.

(c) (40 %)

- U is convex by (a) and $u(x^*)$ is a boundary point of U by (b). By the supporting hyperplane theorem, there exists $\lambda (\neq 0) \in \mathbb{R}^I$ such that $\lambda \cdot u(x^*) \geq \lambda \cdot u$ for all $u \in U$.
- Each component of λ must be nonnegative (hence $\lambda > 0$) because U is unbounded below for each component.
- Since $u(x) \in U$ for any $(x, y) \in A$, we have $\lambda \cdot u(x^*) \geq \lambda \cdot u(x)$ for any $(x, y) \in A$. Hence x^* is a solution of the above optimization problem.

2. Existence of Competitive Equilibrium

Consider a pure exchange economy $\mathcal{E}^{pure} = \left(\left\{\mathbb{R}^L_+, \succeq_i, e_i\right\}_{i=1,...,I}\right)$ (with free disposal) where $\succeq_i$ is continuous, monotone and strictly convex, and $e_i \gg 0$ for each i. Suppose that the market is regulated in this economy and there is a limit on the amount of each good which a consumer can consume. More specifically, let $K_\ell > 0$ be the limit per consumer for good ℓ for $\ell = 1, ..., L$. Assume that $K = (K_1, ..., K_L) \geq e_i$ for all i.

Given this restriction, consumer i's problem becomes

$$\max_{x_i \in \mathbb{R}^L_+} u_i(x_i) \text{ s.t. } p \cdot x_i \leq p \cdot e_i \text{ and } x_{i,\ell} \leq K_\ell \text{ for all } \ell,$$

where u_i is consumer i's utility function that represents $\succeq_i$. $(x^*, p^*) \in \mathbb{R}^{L\times I}_+ \times \mathbb{R}^L_+$ is a **competitive equilibrium** if (1) x^*_i is a solution for the above problem given p^* for each i and (2) $\sum_{i=1}^I x^*_i \leq \sum_{i=1}^I e_i$, where $\sum_{i=1}^I x^*_{i,\ell} = \sum_{i=1}^I e_{i,\ell}$ if $p^*_\ell > 0$ for any ℓ.

(a) Explain what is Walras' law and why it is satisfied in this economy even with this additional restriction on consumption.

(b) Show that there exists a competitive equilibrium in the above sense in this economy given any such $K \gg 0$.

(c) Show that a competitive equilibrium given K is in fact a genuine competitive equilibrium for the pure exchange economy $\mathcal{E}^{pure}$ without the consumption restriction if $K = (K_1, ..., K_L)$ is large enough.

Answer for Q2:

(a) (30 %).

- Walras' law means that each consumer's budget constraint is binding, i.e. $p \cdot x_i(p) = p \cdot e_i$, where $x_i(p)$ is any solution for consumer i's utility maximization problem (or its consequence $p \cdot \sum_{i=1}^{I} x_i(p) = p \cdot r$ is sometimes called Walras' law, which is an OK answer).
- Since $\succeq_i$ is continuous, we can take u_i to be continuous. $\succeq_i$ is strictly convex if and only if u_i is strictly quasi-concave, so $x_i(p)$ is a well defined function on $\mathbb{R}_+^L$ (because the feasible set is compact due to the upper bound K even when some prices are 0). In addition, u_i is strictly increasing because monotonicity + strict convexity $\Rightarrow$ strong monotonicity of $\succeq_i$
- WL is trivially satisfied for $p = 0$, so take any price $p > 0$ and suppose that $p \cdot x_i(p) < p \cdot e_i$. Since $p \cdot e_i \leq p \cdot K$, $x_{i,\ell}(p) < K_\ell$ for some ℓ. This is a contradiction because u_i is strictly increasing and $x_i(p)$ is supposed to be the optimal consumption vector for consumer i given p.

(b) (40%)

- Let $z(p) = \sum_{i=1}^{I} x_i(p) - r$ be the excess demand function. $z(p)$ is well defined and continuous on $\mathbb{R}_+^L$ because $x_i(p)$ is continuous by the Maximum theorem (Note: The budget set must be a continuous correspondence with respect to $p \in \mathbb{R}_+^L$ to apply MT. This follows from the bound K and $e_i \gg 0$).
- Define the following function $f : \triangle \to \triangle$, where $\triangle$ is the price simplex.

  $$f_\ell(p) = \frac{p_\ell + \max\{z_\ell(p), 0\}}{1 + \sum_{\ell=1}^{L} \max\{z_\ell(p), 0\}}$$

  Since f is a continuous function $\triangle$ to $\triangle$, there exists a fixed point p^* s.t. $f(p^*) = p^*$ by Brouwer's fixed point theorem.
- Multiplying the above equation for good ℓ by $z_\ell(p^*)$, summing them up with respect to ℓ and applying Walras' law, $z(p^*) \leq 0$ is obtained.
- Using Walras' law once again, $p^* \cdot z(p^*) = 0$. Hence $z_\ell(p^*) = 0$ if $p_\ell^* > 0$. Therefore (x^*, p^*) is a competitive equilibrium.

(c) (30 %)

- We just need to show that $x_i\left(p^*\right) = x_i^*$ is a solution for consumer i's utility maximization problem even without the bound K.

- Set K large enough so that $K \gg r$. Since $z(p^*) \le 0 \Leftrightarrow \sum_{i=1}^{I} x_i\left(p^*\right) \le r$ (and $X_i = \mathbb{R}_+^L$), $x_i\left(p^*\right) \le r \ll K$, i.e. $x_i \le K$ is not binding for any good for any consumer i.

- Suppose that $x_i\left(p^*\right) = x_i^*$ is not a solution without $x_i \le K$. Then there exists x_i' such that $p \cdot x_i' \le p \cdot e_i$ and $u_i\left(x_i'\right) > u_i\left(x_i^*\right)$. Then we obtain (1) $u_i\left(\alpha x_i' + (1-\alpha)\, x_i^*\right) > u_i\left(x_i^*\right)$ (by strict quasi-concavity) and $p \cdot \left(\alpha x_i' + (1-\alpha)\, x_i^*\right) \le p \cdot e_i$ for any $\alpha \in (0,1)$ and (2) $\alpha x_i' + (1-\alpha)\, x_i^* \ll K$ for small enough $\alpha > 0$. This is a contradiction to the fact that $x_i\left(p^*\right) = x_i^*$ is an optimal solution for consumer i's utility maximization problem with $x_i \le K$.

1. Battle Royal: Cournot vs. Stackelberg:[1] Consider the following game.

		2: s	c
1	S	5 , 2	3 , 1
	C	6 , 3	4 , 4

(a) Suppose the players move simultaneously. What are the Nash equilibria?

(b) Suppose Player 1 first chooses $a_1 \in \{S, C\}$, Player 2 sees his action and then chooses $a_2 \in \{s, c\}$. What are the SPNE?

Suppose Player 1 moves first, but that Player 2 observes 1's action with noise. In particular, Player 2 sees signal $\phi \in \{C, S\}$ such that

$$\Pr(\phi = S | a_1 = S) = 1 - \epsilon \qquad \text{and} \qquad \Pr(\phi = C | a_1 = C) = 1 - \epsilon.$$

(c) Draw the extensive form of the game.

(d) Suppose $\epsilon = 0$. What are the pure strategy weak-PBE? Are any of these sequential equilibria?

(e) Suppose $\epsilon \in (0, 1/4)$. What are the pure strategy weak-PBEs? Explain the difference between this and the last answer.

(f) Now consider mixed strategy equilibria. Suppose Player 1 plays $a_1 = S$ with probability $\lambda \in (0, 1)$. Suppose Player 2 plays s with probability $\eta(C) \in (0, 1)$ after signal $\phi = C$, and plays s with probability $\eta(S) = 1$ after signal $\phi = S$. Let $\mu(\phi)$ be Player 2's belief that $a_1 = S$ after signal $\phi \in \{C, S\}$. Find the mixed strategy equilibrium.

Solution:

(a) Player 1 has a strictly dominant strategy so (C,c) is the unique NE. Note 'C' stands for Cournot.

(b) (S,s) is the unique SPNE. Note 'S' stands for Stackelberg.

(c) Best to draw a "beer-quiche" style extensive form, with Player 1 choosing left/right, the signal going up/down, and then player 2 choosing.

[1] Source: Bagwell (1995, GEB)

(d) (S,s) and (C,c) are weak-PBEs. For (S,s) we need Player 2 to play s after $\phi = S$ and c after $\phi = C$. For (C,c) we need Player 2 to play c no matter which signal occurs. Only (S,s) is sequential. If Player 1 is playing a totally mixed strategy then Player 2 should play s after signal $\phi = S$.

(e) (C,c) is the only pure weak-PBE (or NE). Under (S,s) then player 1 should defect. In part (d) we trembled the Player 1's strategy, so Player 2 should believe the signal off the equilibrium path. Here we tremble Player 2's signal, so in a pure strategy equilibrium the signal is completely uninformative. The noise in the signal thus means that Player 1 can defect without being observed, which makes the equilibrium impossible to sustain.

(f) Player 1's indifference condition is:

$$\begin{aligned} 5(1-\epsilon) + [5\eta(C) + 3(1-\eta(C))]\epsilon &= 6\epsilon + [6\eta(C) + 4(1-\eta(C))](1-\epsilon) \\ (1-\varepsilon) - 3\varepsilon &= 2\eta(C)(1-\epsilon) - 2\eta(C)\varepsilon \\ \eta(C) &= \frac{1-4\varepsilon}{1-2\varepsilon} \end{aligned}$$

Player 2's belief μ after signal C is given by Bayes' rule

$$\mu(C) = \frac{\epsilon\lambda}{\epsilon\lambda + (1-\epsilon)(1-\lambda)}$$

Player 2's indifference condition after $\phi = C$ is

$$2\mu(C) + 3(1-\mu(C)) = \mu(C) + 4(1-\mu(C))$$

yielding $\mu(C) = 1/2$. Thus

$$\begin{aligned} \frac{\epsilon\lambda}{\epsilon\lambda + (1-\epsilon)(1-\lambda)} &= \frac{1}{2} \\ 2\epsilon\lambda &= \epsilon\lambda + 1 - \epsilon - \lambda + \epsilon\lambda \\ \lambda &= 1 - \epsilon \end{aligned}$$

2. Electronic Mail Game:[2] Two armies are trying to coordinate an attack. The attack succeeds if both armies attack and the enemy is weak, but it fails if only one of them attacks or the enemy is strong. The payoff matrix is given by

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{2} \\ & & A & N \\ \cline{3-4} 1 & A & x\ ,\ x & -1\ ,\ 0 \\ \cline{3-4} & N & 0\ ,\ -1 & 0\ ,\ 0 \\ \cline{3-4} \end{array}$$

where $x = 1$ if the enemy is weak and $x = -1$ if the enemy is strong.

Assume first that it is common knowledge that the enemy is weak, $x = 1$.

(a) What actions are rationalizable for either army?

Assume next that there is a (small) chance that the enemy is strong, $\Pr(x = -1) = \varepsilon \in (0, 1)$. Army 1 observes x. If $x = 1$, an e-mail is sent to army 2; however the e-mail only arrives with probability $1 - \varepsilon$ (if $x = -1$ no e-mail is sent). If army 2 receives the e-mail, a confirmation e-mail is sent to army 1; again the confirmation e-mail only reaches army 1 with probability $1 - \varepsilon$. Then the armies independently choose whether to attack.

(b) Argue that N is a strictly dominant strategy for army 2 if it does not receive the e-mail.

(c) Argue that N is the unique rationalizable strategy for army 1 if it knows that the enemy is weak, $x = 1$, but does not receive the confirmation e-mail.

(d) Now assume that whenever an army receives a confirmation e-mail, a confirmation e-mail is sent to the other army and arrives with probability $1 - \varepsilon$. After the end of the e-mail exchange (it ends in finite time with probability one) both armies independently choose whether to attack. Argue by induction that N is the unique rationalizable strategy for an army that has received the n^{th} e-mail but not the $n + 2^{th}$ e-mail (n is even for army 1 and odd for army 2).

Solution:

(a) Both actions are rationalizable because N is a best response to N and A is a best response to A.

(b) If army one does not receive the e-mail, there are two possibilities: A) the state is $x = -1$; B) the state is $x = 1$ but the e-mail got lost. Case A has probability ε and case B has probability $(1 - \varepsilon)\varepsilon$, so the conditional probability of case A equals $p = \frac{\varepsilon}{\varepsilon + (1-\varepsilon)\varepsilon} > \frac{1}{2}$. So even if army 1 attacks, the utility from attacking for army 2 equals $p(-1) + (1 - p)1 < 0$.

(c) If army 1 observes $x = 1$ but does not receive the confirmation e-mail there are two possibilities: A) the initial e-mail got lost; B) the initial e-mail arrived but the confirmation e-mail got lost. Conditionally only on $x = 1$, case A has probability ε and case B has probability $(1 - \varepsilon)\varepsilon$, so conditionally on army 1 not receiving the confirmation e-mail the probability of

[2] Source: Rubinstein (1989, AER)

case A equals $p = \frac{\varepsilon}{\varepsilon+(1-\varepsilon)\varepsilon} > \frac{1}{2}$. By part (b) we know that army 2 does not attack in case A, so army 1's expected utility from attacking is at most $p(-1) + (1-p)1 < 0$.

(d) Parts (b) shows that the statement is true for $n = 0$. Assume that the statement is true up to some $n - 1 \geq 0$, so N is the unique rationalizable strategy for an army that has received the $n-1^{th}$ e-mail, but not the $n+1^{th}$. Now consider the army that has received the n^{th} e-mail but not the $n+2^{th}$. Again there are two possibilities: Either the $n+1^{th}$ e-mail got lost, or the $n+2^{th}$ e-mail got lost. Again, conditional on the n^{th} e-mail having arrived the conditional probability that the $n+1^{th}$ e-mail got lost is $p = \frac{\varepsilon}{\varepsilon+(1-\varepsilon)\varepsilon} > 1/2$. In that case, the other army must rationalizably choose N by induction. But then the expected utility from attacking is at most $p(-1) + (1-p)1 < 0$, proving the induction step.