UCLA

Department of Economics

Ph. D. Preliminary Exam

Micro-Economic Theory

(SPRING 2012)

Instructions: You have 4 hours for the exam. Answer any 5 out of the 6 questions. All questions are weighted equally. Answering fewer than 5 questions is not advisable, so do not spend too much time on any question. Do NOT answer all questions. Use a SEPARATE bluebook to answer each question.

(1) **(Competition in TU economy)** Consider an exchange economy with one money good m and two consumption goods x, y. Holdings of m can be positive, negative or zero; holdings of x, y must be non-negative. There are $M \geq 1$ agents of Type 1 and $N \geq 1$ agents of Type 2; they have identical utility functions $u(m, x, y) = m + x^{1/2}y^{1/2}$ but different endowments: $e_1 = (0, 1, 0)$, $e_2 = (0, 0, 1)$. (In words: agents of Type 1 are endowed with 1 unit of x, agents of Type 2 are endowed with 1 unit of y; no one is endowed with money. Remember that holdings of money can be positive, negative or zero.) Normalize so that the price of money $p_m = 1$.

(a) Find the Pareto optimal allocations

(b) Verify that the unique (normalized) supporting price is the same for all Pareto optimal allocations:

$$p_m = 1\ ,\ p_x = (1/2)(N/M)^{1/2}\ ,\ p_y = (1/2)(M/N)^{1/2}$$

(c) Given the supporting prices as in (b), find the unique Walrasian equilibrium consumptions and utilities (for each agent of Type 1 and for each agent of Type 2)

(d) Find the social gains function $g(M, N)$ (the maximal utility obtainable by society) as a function of the numbers M, N of agents of each type

(e) In this context, we can define the marginal contributions of each agent of Type 1 and each agent of Type 2 as

$$\begin{aligned} MC_1 &= g(M,N) - g(M-1,N) \\ MC_2 &= g(M,N) - g(M,N-1) \end{aligned}$$

Suppose $M, N \to \infty$ in such a way that M/N and N/M remain bounded. Show that the difference between the marginal contribution of each agent and the equilibrium utility of that agent converges to 0. (Suggestion: use a first degree Taylor polynomial to approximate marginal contributions. Complete rigor not required.)

2. Discriminating monopoly

Novartis holds the patent and is the sole producer of a new drug for the treatment of a rare disease. The marginal cost of producing the drug is $c = 20$. The drug is available in the US (country 1) and Mexico (country 2.) The respective demand functions are

$q_1 = Max\{0, 600 - 5p_1\}$ and $q_2 = Max\{0, 75 - \frac{5}{4}p_2\}$.

(Mexico is a smaller country than the US and the disease is relatively rarer in the US.)

(a) Solve for the profit maximizing quantities, prices and profits in the two countries.

(b) If the manufacturer is prohibited from selling in different countries at different prices, what quantity will the firm produce, what price will it charge and what profit will it make?

(c) How would your answer to (b) change if the marginal cost were to rise? HINT: Try appealing to the Envelope Theorem.

3. Stick and Carrot

Consider a two-player repeated game with the following stage game.

	C	D	E
C	$1, 1$	$-1, 2$	$-x+1, -x+1$
D	$2, -1$	$0, 0$	$-x+1, -x+1$
E	$-x+1, -x+1$	$-x+1, -x+1$	$-x, -x$

The players discount their future payoffs by discount factor $\delta \in (0, 1)$. Answer the following questions.

(a) Suppose that $x = 1$. Characterize the range of δ in which there exists a subgame perfect equilibrium (SPE) that is fully cooperative (i.e. the equilibrium outcome is $((C, C), (C, C), \ldots)$).

(b) Suppose that $x = 3$. Assume that a public randomization device is available. Characterize the range of δ in which there exists a fully cooperative SPE.

(c) Suppose that $\delta = \frac{1}{3}$. Assume that a public randomization device is available. Characterize the range of $x \geq 0$ in which there exists a fully cooperative SPE.

4. Rationalizability in a Location Game

Two restaurants α and β are about to open on the same street, which is modeled as closed interval $[0, 1]$. They decide simultaneously where to locate on $[0, 1]$. They can choose any $\frac{k}{n}$ for $k = 0, 1, ..., n$ where $n > 1$ is an even integer. Suppose that consumers are distributed uniformly on $[0, 1]$. Once two restaurants open, each consumer will go to the closer one (if the restaurants choose the same location, then a consumer choose each restaurant with equal probability, hence each restaurant gets the half of consumers). Restaurant i's profit is proportional to the number of the consumers who choose restaurant i. The objective of each restaurant is to maximize its expected profit. Answer the following questions.

(a) Describe this situation formally as a strategic game.

(b) Find all rationalizable actions for each restaurant and all Nash equilibria in this game.

(c) Add one more restaurant γ. Suppose that $n = 8$. Find all rationalizable actions for each restaurant.

5. Sealed-bid auctions

A single item is to be sold by sealed bid auction. Let b_H be the highest bid submitted and let b_S be the second highest bid submitted. The high bidder wins and pays $\alpha b_H + (1-\alpha)b_S$ where $\alpha \in [0,1]$. There are *n* buyers. Buyer *i* , $i = 1,...,n$ has a value θ_i which is an independent draw from a distribution with support $[0,1]$, c.d.f $F(\theta)$ and p.d.f. $f(\theta)$.

(a) Solve for the equilibrium expected payoff $V(\theta)$. As far as possible prove every claim that you make.

(b) Draw a conclusion as to the effect on both buyer's expected payoffs and expected seller revenue as the parameter α varies.

(c) The seller announces a reserve price (minimum bid) of $\hat{\theta}$. If there is only one bidder, that bidder will pay $\alpha b + (1-\alpha)\hat{\theta}$. Do the conclusions of (b) continue to hold? Explain carefully.

(d) If $n = 2$ and $F(\theta) = \theta$ obtain a differential equation for the equilibrium bid function. Confirm that for some *k*, $B(\theta) = k\theta$ is an equilibrium bid function.

6. Efficient mechanism design

Let $B(\theta_i, q)$ be the benefit to agent i, $i = 1, 2$ if the designer chooses q and the agent's type is $\theta_i \in \Theta_i = [\alpha_i, \beta_i]$. The cost of this action to the designer is $C(q)$. Let $S(\theta_1, \theta_2, q)$ be social surplus if the designer chooses q and let $q^*(\theta)$ be the social surplus maximizing choice.

(a) Explain why $\theta_1 = \arg\underset{x_1}{Max}\{S(\theta_1, x_2, q^*(x_1, x_2))$.

(b) The designer offers a transfer payment $t_i(x)$ to each agent and commits to choosing $q^*(x)$ based on the agents' responses. What is the transfer payment with the property that it is a dominant strategy for agent i to reveal his true type?

(c) Show that if the designer chooses transfer payments such that the participation constraint is binding, then the designer's payoff is

$U^D(\theta) = S^*(\alpha_1, \theta_2) + S^*(\theta_1, \alpha_2) - S^*(\theta_1, \theta_2)$ where $S^*(\theta) \equiv S(\theta, q^*(\theta))$.

(d) Suppose that some amount q of a commodity is to be produced for agent 1 with benefit function $B(\theta_1, q) = (4 + \theta_1)q - \frac{1}{2}q^2$. The commodity will be produced by agent 2. Only she knows the constant unit cost $c = 2 - \theta_2$. Both θ_1 and $\theta_2 \in [0,1]$. Show that $q^*(\theta) = (4 + \theta_1 - c) = (2 + \theta_1 + \theta_2)$ and that maximized social surplus is $S^*(\theta) = \frac{1}{2}(2 + \theta_1 + \theta_2)^2$. Then appeal to (c) to examine whether the mechanism is profitable for the designer.

ANSWERS

ANSWER to 1

(a) Because this is a TU economy, Pareto optimality is obtained by maximizing social welfare (sum of all utilities). This boils down to maximizing the sum of utilities for consumption and distributing money arbitrarily. Because utility for consumption is identical across consumers and strictly concave, the sum of utilities is maximized by

distributing the total endowment of consumption goods $(0, M, N)$ equally across $M + N$ consumers. Hence the Pareto optimal allocations are

$$\left\{(m_t, M/(M+N), N/(M+N)) : \sum m_t = 0\right\}$$

(b) The supporting price equalizes marginal utilities per dollar at the Pareto optimal allocation(s). Marginal utilities are

$$\begin{aligned} MU_x &= (1/2)x^{-1/2}y^{1/2} \\ MU_y &= (1/2)x^{1/2}y^{-1/2} \end{aligned}$$

Plugging in $x = M/(M+N)$, $y = N/(M+N)$ and noticing that the denominators $(M+N)$ cancel gives

$$\begin{aligned} p_x &= (1/2)(N/M)^{1/2} \\ p_y &= (1/2)(M/N)^{1/2} \end{aligned}$$

(c) Given the supporting prices as in (b), the unique WE is determined by the budget constraint. For agents of Type 1 this is

$$m_1+(1/2)[(N/M)^{1/2}][M/(M+N)]+(1/2)[(M/N)^{1/2}][N/(M+N)] = (1/2)[(N/M)^{1/2}]$$

Solving and simplifying gives

$$\begin{aligned} m_1 &= (1/2)[(N/M)^{1/2} - (M/N)^{1/2}][N/(M+N)] \\ u_1 &= [(N/M)^{1/2} - (M/N)^{1/2}][N/(M+N)] + (MN)^{1/2}/(M+N) \end{aligned}$$

Similarly

$$\begin{aligned} m_2 &= [(M/N)^{1/2}] - (N/M)^{1/2}][M/(M+N)] \\ u_2 &= [(M/N)^{1/2}] - (N/M)^{1/2}][M/(M+N)] + (MN)^{1/2}/(M+N) \end{aligned}$$

(d) $g(M,N) = M[M^{1/2}N^{1/2}/(M+N)] + N[M^{1/2}N^{1/2}/(M+N)] = M^{1/2}N^{1/2}$

(e) $MC_1 = g(M,N) - g(M-1,N) = [M^{1/2} - (M-1)^{1/2}]N^{1/2}$
$MC_2 = g(M,N) - g(M,N-1) = M^{1/2}[N^{1/2} - (N-1)^{1/2}]$

ANSWER to 2

The demand price functions are $p_1 = 120 - \frac{1}{5}q_1$ and $p_2 = 60 - \frac{4}{5}q_2$. Then

$MR_1 - c = 100 - \frac{2}{5}q_1 = 0$ at $q_1^* = 250$ $MR_2 - c = 40 - \frac{8}{5}q_2$ so $q_2^* = 25$.

Then $p_1 = 70$, $\Pi_1 = (p_1 - c)q_1 = 12{,}500$, $p_2 = 40$ and $\Pi_2 = (p_2 - c)q_2 = 500$.

(b) For $p \geq 60$, see answer to (a). For $p < 60$ $q = q_1 + q_2 = 675 - \frac{25}{4}p$. Then $p = 108 - \frac{4}{25}q$.

So $MR = 108 - \frac{8}{25}q$. $MR - c = 88 - \frac{8}{25}q$ so $q = 275$ and $p = 64$. The $\Pi = 44 * 275 = 12100$.

This is less than Π_1^* in (a)

(c) Let $\Pi_U(c)$ be maximized profit if only the US is served. Let $\Pi_B(c)$ be the profit if only the US is served. $\Pi(c) = \underset{q}{Max}\{(p(q) - c)q\}$. Therefore by the Envelope Theorem $\Pi'(c) = -q(c)$.
Since sales are higher at the lower price when both are served it follows that

$$\Pi_U{}'(c) = -q_u(c) > -q_B(c) = \Pi_B{}'(c).$$

Therefore the profit difference is even higher if marginal cost rises.

ANSWER to 3

(a) The minmax payoff is 0 in this case. So we can use the Nash reversion to support any equilibrium without loss of generality.

Hence a fully cooperative SPE exists if and only if

$$\begin{aligned} 1 &\geq (1-\delta)2 + \delta \times 0 \\ &\Updownarrow \\ \delta &\geq \frac{1}{2} \end{aligned}$$

(b) The minmax payoff is -2 in this case. Since no SPE payoff can be below -2, the following condition is necessary for any fully cooperative SPE to exist.

$$\begin{aligned} 1 &\geq (1-\delta)2 + \delta(-2) \\ &\Updownarrow \\ \delta &\geq \frac{1}{4} \end{aligned}$$

Next we show that this condition is in fact sufficient: we can construct a fully cooperative strongly symmetric SPE with a public randomization device for any such δ.

Consider the following strategy (starting in Phase 1):

Phase 1: Play C. Move to Phase 2 if and only if there is a unilateral deviation from (C, C).

Phase 2: Play E. If (E, E) is played, then move back to Phase 1 with probability ρ and stay in Phase 2 with probability $1-\rho$ by using a public randomization device. Otherwise stay in Phase 2 with probability 1, where

$$\rho := \frac{1}{3}\left(\frac{1}{\delta} - 1\right)$$

which is in $[0, 1]$ for any $\delta \geq \frac{1}{4}$.

Note: ρ is chosen to satisfy the following equation

$$(1-\delta)(-3) + \delta\left[(1-\rho)(-2) + \rho\right] = -2$$

, which means that the continuation payoff in Phase 2 is -2.

By construction the one shot deviation constraints in Phase 1 and Phase 2 are satisfied. Hence this symmetric strategy profile constitutes a fully cooperative SPE.

Therefore, a fully cooperative SPE exists if and only if $\delta \in [\frac{1}{4}, 1)$.

(c) Let $\underline{v}$ be the minmax payoff. To support a fully cooperative, $\underline{v}$ must satisfy

$$1 \geq (1-\delta)2 + \delta\underline{v},$$

that is

$$\underline{v} \leq \frac{2\delta - 1}{\delta} = -1.$$

So the minmax payoff must be at most -1.

The minmax payoff is less than or equal to -1 if and only if $x \geq 2$. Hence $x \geq 2$ is necessary. On the other hand, it is easy to verify that the "stick and carrot" equilibrium used in (b) supports a fully cooperative equilibrium whenever $x \geq 2$ given $\delta = \frac{1}{3}$ (with appropriate choice of ρ).[1] Hence the desired range of x is $x \geq 2$.

ANSWER TO 4

(a) The set of players is $\{\alpha, \beta\}$. Player i's action set is $A_i = \left\{0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\right\}, i = \alpha, \beta$. Player i's (normalized) payoff is as follows:

$$u_i(a) = \begin{cases} \frac{a_i + a_{-i}}{2} & \text{if } a_i < a_{-i} \\ \frac{1}{2} & \text{if } a_i = a_{-i} \\ 1 - \frac{a_i + a_{-i}}{2} & \text{if } a_i > a_{-i} \end{cases}$$

(b) A set of rationalizable actions can be obtained by eliminating strictly dominated actions iteratively. In the first round, 0 is strictly dominated by $\frac{1}{n}$. Similarly 1 is strictly dominated by $\frac{n-1}{n}$. All the other locations are not strictly dominated: $\frac{k}{n}$ is a strict best response to $\frac{k-1}{n}$ for $k = 1, ..., \frac{n}{2}$ and a strict best response to $\frac{k+1}{n}$ for $k = \frac{n}{2}, ..., n-1$. In the second round, $\frac{1}{n}$ and $\frac{n-1}{n}$ are the only locations that are strictly dominated, hence eliminated. More generally, $\frac{k-1}{n}$ and $\frac{n-k+1}{n}$ are eliminated in the kth round of elimination. Since n is an even integer, the only location that survives such iterated eliminations is $\frac{n}{2}$. Hence $\frac{n}{2}$ is the only rationalizable action for each restaurant.

Since the actions eliminated above are never played in any Nash equilibrium (whether pure or mixed), the only candidate of Nash equilibrium is $\left(\frac{n}{2}, \frac{n}{2}\right)$. Since there exists a Nash equilibrium for any finite strategic game, $\left(\frac{n}{2}, \frac{n}{2}\right)$ must be the only Nash equilibrium (or this can be directly verified).

(c) Again delete strictly dominated actions iteratively. In the first round, it is easy to show that 0 and 1 are strictly dominated by $\frac{1}{8}$ and $\frac{7}{8}$ respectively. All the other locations are not strictly dominated: $\frac{k}{n}$ is a strict best response to (the other two players' choosing) $\frac{k-1}{n}$ for $k = 1, ..., 4$ and a strict best response to $\frac{k+1}{n}$ for $k = 4, 5, 6, 7$. In the second round, we can show that $\frac{1}{8}$ and $\frac{7}{8}$ are never best response (which is equivalent to being strictly dominated by some mixed strategy). For example, take $\frac{1}{8}$. If $a_j > \frac{1}{4}$ for every $j \neq i$, then $\frac{1}{8}$ is clearly not a best response ($\frac{1}{4}$ is better). Suppose that there exists $j \neq i$ such that $a_j = \frac{1}{8}$ or $\frac{1}{4}$. Consider two cases: $a_k < \frac{3}{4}$ or $a_k \geqq \frac{3}{4}$ for $k \neq i, j$. In the former case, there is a profitable deviation to $\frac{3}{4}$, which guarantees at least the profit of $\frac{1}{4}$. In the latter case, $\frac{1}{4}$ is better than $\frac{1}{8}$. Hence $\frac{1}{8}$ is never a best response. Similarly $\frac{7}{8}$ is never a best response.

Finally, $\left\{\frac{1}{4}, \frac{3}{8}, ..., \frac{5}{8}, \frac{3}{4}\right\}$ is the set of rationalizable actions because every location in it is a best response to the other two restaurants choosing $\frac{1}{4}$ and $\frac{3}{4}$ (hence no more location can be eliminated).

ANSWER to 5

(a) If all other buyers bid according to $B(\theta)$ and buyer 1 bids $B(x)$ she wins if $\theta_j < x,\ \ j = 2,...,n$. Then her win probability is

$$w(x) = F^{n-1}(x)$$

Her expected payoff is therefore

$u(\theta, x) = w(x)\theta - r(x)$ where $r(\theta)$ is the equilibrium expected payment.

For equilibrium,

$$\theta_1 = \arg \underset{x}{Max}\{u(\theta, x) = w(x)\theta - r(x)\,.$$

Let $V(\theta)$ be the equilibrium payoff. Then

$$V(\theta) = w(\theta)\theta - r(\theta) \qquad (*)$$

and by the Envelope Theorem,

$$V'(\theta_1) = u_\theta(\theta_1, x)\ \Big|_{x=\theta_1} = w(\theta_1) = F^{n-1}(\theta_1)\,.$$

Then

$$V(\theta) = \int_0^\theta F^{n-1}(\theta_1)d\theta_1\,. \qquad (**)$$

(b) Appealing to (**) $V(\theta)$ is independent of the parameters thus the buyer's equilibrium payoffs are unaffected. By (*) the expected payment by a buyer is unaffected. Thus expected revenue is unaffected.

(c) With the reserve price entry if and only if $\theta \geq \hat{\theta}$. Then $V(\hat{\theta}) = 0$ and so arguing as before

$$V(\theta) = V(\theta) - V(\hat{\theta}) = \int_{\hat{\theta}}^{\theta} F^{n-1}(\theta_1)d\theta_1$$

(d) $V(\theta) = \int_0^\theta w(x)dx = \int_0^\theta F(x)dx = \int_0^\theta xdx = \tfrac{1}{2}\theta^2\,.$

Also $V(\theta) = F(\theta)\theta - r(\theta)$. Therefore $r(\theta) = \tfrac{1}{2}\theta^2$

$$r(\theta)=\int_0^\theta [\alpha B(\theta)+(1-\alpha)B(x)]f(x)dx=\alpha B(\theta)\theta+(1-\alpha)\int_0^\theta B(x)dx$$

Then

$r'(\theta)=\alpha B'(\theta)\theta+B(\theta)$.

But $r(\theta)=\frac{1}{2}\theta^2$.

Therefore

$\alpha B'(\theta)\theta+B(\theta)=\theta$.

Try $B(\theta)=k\theta$

$\alpha B'(\theta)\theta+B(\theta)=\alpha k\theta+k\theta=(\alpha+1)k\theta$

Therefore the ODE holds if $k=\dfrac{1}{1+\alpha}$

ANSWER TO 6

(a) Since $q^*(\theta_1,\theta_2)=\arg\underset{\theta}{Max}\,S(\theta_1,\theta_2,q)$ it follows that

$$S(\theta_1,\theta_2,q^*(x_1,\theta_2))\le S(\theta_1,\theta_2,q^*(\theta_1,\theta_2)) \text{ for all } \theta_2\in\Theta_2 .$$

Changing notation,

$S(\theta_1,x_2,q^*(x_1,x_2))\le S(\theta_1,x_2,q^*(\theta_1,x_2))$ for all $x_2\in\Theta_2$.

Therefore

$$\theta_1=\arg\underset{x_1}{Max}\{S(\theta_1,x_2,q^*(x_1,x_2))\} \text{ for all } x_2\in\Theta_2 . \qquad (*)$$

(b) Thus truth telling is a dominant strategy if agent 1's payoff is $S(\theta_1,x_2,q^*(x_1,x_2))$

$$S(\theta_1,x_2,q^*(x_1,x_2))=B(\theta_1,q^*(x))+B(x_2,q^*(x))-C(q^*(x)$$

Then choose $t_1(x)=B(x_2,q^*(x))-C(q^*(x))$

(c) Given this transfer agent 1's equilibrium payoff is $S^*(\theta)=S(\theta,q^*(\theta))$

Subtract off the minimum payoff so that the participant constraint is binding.

Then $U_1 = S^*(\theta_1,\theta_2) - S(\alpha_1,\theta_2)$

By the same argument, $U_2 = S^*(\theta_1,\theta_2) - S(\theta_1,\alpha_2)$

The sum of the payments must equal social surplus. Then

$$S^*(\theta_1,\theta_2) = U_1 + U_2 + U^D = S^*(\theta_1,\theta_2) - S(\alpha_1,\theta_2) + S^*(\theta_1,\theta_2) - S(\theta_1,\alpha_2) + U^D$$

Social surplus is

$$S(\theta_1,\theta_2,q) = (4+\theta_1)q - \tfrac{1}{2}q^2 - (2-\theta_2)q = (2+\theta_1+\theta_2)q - \tfrac{1}{2}q^2$$

Then $q^*(\theta) = (2+\theta_1+\theta_2)$ and so $S^*(\theta) = \tfrac{1}{2}(2+\theta_1+\theta_2)^2$.

From (c) it follows that

$$U^D(\theta) = \tfrac{1}{2}[(2+\theta_2)^2 + (2+\theta_1)^2 - (2+\theta_1+\theta_2)^2]$$

This is positive at $\theta = (0,0)$ and $\theta = (1,1)$.

A perfect answer would also show that $U^D(\theta)$ is a decreasing function.

$$\frac{\partial U^D}{\partial \theta_1} = (2+\theta_1) - (2+\theta_1+\theta_2) = -\theta_2 < 0,\ \ \theta_1 \in (0,1]$$