UCLA

Department of Economics

Ph. D. Preliminary Exam

Micro-Economic Theory

(SPRING 2010)

Instructions:

- You have **4** hours for the exam
- Answer any **5** out of the **6** questions. All questions are weighted equally. Answering fewer than **5** questions is not advisable, so do not spend too much time on any question. Do NOT answer all questions.
- Use a SEPARATE bluebook to answer each question.

1. Expected Utility

Let $\triangle(Z)$ be the space of lotteries on a finite set Z. Let $\succeq$ be a complete and transitive preference on $\triangle(Z)$. Consider the following version of independence axiom:

$$\begin{aligned} \text{(IA)} \quad &: \quad \text{For any } p, q, r \in \triangle(Z) \text{ and } a \in (0,1), \\ p \quad &\succ \quad q \Rightarrow ap + (1-a)r \succ aq + (1-a)r. \end{aligned}$$

Answer the following questions.

(a) Show that, if $\succeq$ satisfies (IA), then it satisfies the following:

$$\begin{aligned} \text{For any } p, q \quad &\in \quad \triangle(Z) \text{ and } a, b \in [0,1], \\ p \quad &\succ \quad q \;\&\; a > b \Rightarrow ap + (1-a)q \succ bp + (1-b)q. \end{aligned}$$

(b) Suppose that $Z = \{z_1, z_2\}$, i.e. there are only two possible outcomes. Abusing notations, denote the probability of z_1 as p for lottery $p \in \triangle(Z)$. Suppose that (IA) is satisfied. Can you find $u : Z \to \mathbb{R}$ such that $p \succeq q$ if and only if $pu(z_1) + (1-p)u(z_2) \geq qu(z_1) + (1-q)u(z_2)$? If you think that the answer is yes, prove it. If you think no, then describe a counterexample.

2. First Welfare Theorem and Externality

Consider a pure exchange economy with two consumers, who are endowed with strictly positive initial endowments $e_i \in \mathbb{R}^L_{++}, i = 1, 2$. Consumer 1's utility function $u_1 : \mathbb{R}^L_+ \to \mathbb{R}$ is differentiable, concave and satisfies $Du_1(x_1) \gg 0$ for any $x_1 \in \mathbb{R}^L_+$. Assume that consumer 2 cares about consumer 1's consumption of good 1 as well as his or her own consumption. More specifically, consumer 2's utility function $u_2 : \mathbb{R}^{L+1}_+ \to \mathbb{R}$ is given by $u_2(x_2, x_{1,1}) = v_2(x_2) - x_{1,1}$, where v_2 is differentiable, concave and satisfies $Dv_2(x_2) \gg 0$ for any $x_2 \in \mathbb{R}^L_+$.

Pareto efficiency is defined in the standard way. A feasible allocation $(x_1, x_2) \in \mathbb{R}^{2L}_+$ $(x_1 + x_2 \leq r = e_1 + e_2)$ is Pareto efficient if and only if there is no other feasible allocation (x'_1, x'_2) such that $u_1(x'_1) \geqq u_1(x_1)$ and $u_2\left(x'_2, x'_{1,1}\right) \geqq u_2(x_2, x_{1,1})$ with at least one inequality being strict. Answer the following questions.

(a) Consider the following programming problem:

$$\text{(P)} \quad \max_{(x_1,x_2)\in\mathbb{R}^{2L}_+} u_1(x_1) \ \text{ s.t. } v_2(x_2) - x_{1,1} \geq \overline{u}_2 \text{ and } x_1 + x_2 \leq r$$

Show that a feasible allocation $(x^*_1, x^*_2) \in \mathbb{R}^{2L}_+$ is Pareto efficient if and only it is a solution of problem (P) with $\overline{u}_2 = v_2(x^*_2) - x^*_{1,1}$.

(b) Write down the Kuhn-Tucker conditions to characterize the set of all solutions in $\mathbb{R}^{2L}_{++}$ of (P). Discuss why they characterize the entire set of (interior) solutions.

(c) Define competitive equilibrium in this pure exchange economy as follows: $(x^*, p^*) \in \mathbb{R}^{2L}_+ \times \mathbb{R}^L_+$ is a competitive equilibrium if (1) x^*_1 solves $\max_{x_1 \in B(p^*, p^* \cdot e_1)} u_1(x_1)$, (2) x^*_2 solves $\max_{x_2 \in \mathbb{B}(p^*, p^* \cdot e_2)} u_2\left(x_2, x^*_{1,1}\right)$ and (3) $x^*_1 + x^*_2 \leq r$. Use the characterization in (b) to show that no strictly positive competitive equilibrium allocation $x^* \gg 0$ in this economy is Pareto efficient (in particular, the first welfare theorem fails in this economy).

3. Security Markets

Two assets A, B are traded on January 1 of every year; we are interested in the period January 1, 2011 - January 1, 2013. A is risky: the initial price is \$1; each year the price either increases by 20% or decreases by 10%. (That is: on January 1, 2011 the price is either \$1.20 or \$0.90, etc.) B is riskless: the initial price is \$1; the yearly interest rate is 10%.

(a) Find the January 1, 2011 price of a call option on A with exercise date January 1, 2013 and strike price \$1.40.

(b) Find a self-financing trading strategy that replicates this option.

(Assume security prices do not admit arbitrage. To simplify the arithmetic it is OK to round – three decimal digits is fine.)

4. Market Game

There is a single risk-neutral seller and a single risk-neutral buyer, who look to agree to a trade of an asset. The asset's value to the seller is v, while the value to the buyer is αv, for some $\alpha \in (1, 2)$. The value v is uniformly distributed on $[0, V]$.

The buyer and seller can be informed (i.e. know v) or uninformed (i.e. only know the prior distribution of v). The probability that the buyer (seller) is informed is $\varepsilon > 0$.

The game considered is the following: a mediator proposes an arbitrary price p. The buyer and seller then decide whether to agree to trade the object, in which case the buyer pays the seller p in exchange for the object, with the buyer's final payoffs being $\alpha v - p$, and the seller's final payoff being $p - v$. The price p is taken as a parameter, and you will be ask to find the set of prices, at which the two parties agree to trade with some positive probability.

(*i*) Suppose first that the value v is common knowledge among the two parties. For a given p, define the resulting normal form game among the players, and define and characterize its Nash equilibrium. At what prices do the parties agree to trade? Also, discuss how your answer would change if it was common knowledge that neither party knows v, and both share the same common prior.

(*ii*) Suppose next that it is common knowledge that the seller knows v, but not the buyer. For a given p, define the resulting incomplete information game, define and characterize its Bayesian Nash equilibrium. At what prices do the parties agree to trade?

(*iii*) Suppose next that it is common knowledge that the buyer knows v, but not the seller. For a given p, define the resulting incomplete information game, define and characterize its Bayesian Nash equilibrium. At what prices do the parties agree to trade?

(*iv*) Suppose next that the parties know only their own information, but not the other party's. Define the resulting incomplete information game, define and characterize all its Bayesian Nash equilibria.

5. Indirect Price Discrimination

(a) A monopoly offers different "plans" where a plan is a payment r for q units. That is, a q-pack cost r. What is the single crossing property? Confirm that it holds if a type t agent's demand price function $p_t(q)$ is greater for higher types.

(b) Let $\{q_t, r_t\}$ be the plan selected by type t buyers. That is, a type t buyer pays r_t for q_t units. Show that it is necessarily the case that if $s < t$ then $q_s < q_t$.

(c) Prove that for any $\{q_t, r_t\}$ satisfying the above monotonicity condition, revenue is maximized if and only if the local downward constraints are satisfied.

(d) In the two type case suppose that the number of each type is the same ($n_1 = n_2 = n$). Total cost is a strictly convex function $C(nq_1 + nq_2)$. The demand price functions are $p_1 = 100 - \frac{1}{2}q_1$ and $p_2 = 120 - \frac{1}{2}q_2$. If it is profit maximizing to offer two plans, what can you say about the difference in the number of units in each plan?

6. Replica Invariance

Suppose $(\bar{z}_i)$ is a feasible allocation, i.e., $\sum \bar{z}_i = 0$, for the quasilinear model $\mathbf{v} = (v_1, \ldots, v_n)$, where each v_i is merely continuous.

(a) True or false: If there exists a positive integer k and a feasible allocation (z^k_{ih}) for the k-replica of $\mathbf{v}$ such that

$$\sum_i \sum_h v_i(z_{ih}) > k \sum_i v_i(\bar{z}_i),$$

then $(\bar{z}_i)$ cannot be a price-taking equilibrium for $\mathbf{v}$. [z^k_{ih} is the allocation to person h of type i in the k-replica of $\mathbf{v}$.] If true, demonstrate; otherwise, describe a counterexample.

(b) True or false: If for all positive integers k and all feasible allocations (z^k_{ih}), $h = 1, \ldots, k$ for the k-replica of $\mathbf{v}$,

$$\sum_i \sum_h v_i(z_{ih}) \leq k \sum_i v_i(\bar{z}_i),$$

then $(\bar{z}_i)$ is a price-taking equilibrium for $\mathbf{v}$. If true, demonstrate; otherwise, describe a counterexample.

Suppose $(\bar{x}_i)$ is a feasible allocation for the ordinal preferences model of an exchange economy $\mathcal{E} = \{(X_i), (\succeq_i), (\omega_i)\}$, where $X_i = \mathbb{R}^\ell_+$ and each $\succeq_i$ is merely continuous.

(c) True or false: If there exists a positive integer k and a feasible allocation (x^k_{ih}) for the k-replica of $\mathcal{E}$ such that

$$\bar{x}^k_{ih} \succ_i \bar{x}_i \text{ for each } i \text{ and } h = 1, \ldots, k,$$

then $(\bar{x}_i)$ cannot be a price-taking equilibrium for $\mathcal{E}$. If true, demonstrate; otherwise, describe a counterexample.

(d) True or false: Suppose that for every k and every feasible allocation (x^k_{ih}) for the k-replica of $\mathcal{E}$,

$$\bar{x}^k_{ih} \succeq_i \bar{x}_i \text{ for each } i \text{ and } h = 1, \ldots, k \text{ implies } \bar{x}^k_{ih} \sim_i \bar{x}_i.$$

Can you conclude that $(\bar{x}_i)$ is a price-taking equilibrium for $\mathcal{E}$? Explain.