Fall 2016

1. **Stochastic Dominance**

There are three lotteries p_1, p_2, p_3, each of which generates $-\$100$, \$50, or \$200 with the following probability respectively.

$$\begin{aligned} p_1(-100) &= 0.2,\ p_1(50) = 0.4,\ p_1(200) = 0.4 \\ p_2(-100) &= 0.1,\ p_2(50) = 0.5,\ p_2(200) = 0.4 \\ p_3(-100) &= 0,\ p_3(50) = 0.65,\ p_3(200) = 0.35 \end{aligned}$$

Answer the following questions.

(a) Find all pairs of lotteries for which one lottery is first order stochastically dominated by the other. Explain why there is no other such pair.

(b) Find all pairs of lotteries for which one lottery is second order stochastically dominated by the other. Explain why there is no other such pair.

(c) Consider another lottery $\widetilde{p}$ with the probability distribution: $\widetilde{p}(-100) = x$, $\widetilde{p}(50) = y$, $\widetilde{p}(200) = 1 - x - y$. Characterize the range of x and y in which p_1 is first order stochastically dominated by $\widetilde{p}$.

(d) Characterize the range of x and y in which p_1 is second order stochastically dominated by $\widetilde{p}$.

Answer for Q1

(a) (2.5 pt). p is first order stochastically dominated by q if and only if $q(-100) \leq p(-100)$ and $q(-100) + q(50) \leq p(-100) + p(50)$ (which is equivalent to $p(200) \leq q(200)$). So p_1 is first order stochastically dominated by p_2. Other potential pairs are (p_2, p_3) and (p_1, p_3) as $p_3(-100) \leq p_i(-100)$ for $i = 1, 2$. But neither p_1 nor p_2 is first order stochastically dominated by p_3 because $p_3(200) < p_i(200)$ for $i = 1, 2$.

(b) (2.5 pt). p is second order stochastically dominated by q if and only if $q(-100) \leq p(-100)$ and $q(-100) \times 150 + (q(-100) + q(-50)) \times x \leqq p(-100) \times 150 + (p(-100) + p(-50)) \times x$ for all $x \in [0, 150]$. This can be simplified to $q(-100) \leq p(-100)$ and $2q(-100) + q(-50) \leq 2p(-100) + p(-50)$. We already know that p_1 is second order stochastically dominated by p_2 (because FOSD implies SOSD). In addition, according to the above inequalities, p_2 is second order stochastically dominated by p_3. Hence p_1 is second order stochastically dominated by p_3 as well (as SOSD is transitive). There is no other pair because SOSD holds strictly for each pair.

(c) (2.5 pt) p_1 is first order stochastically dominated by $\widetilde{p}$ if and only if $x \leq p_1(-100)$ and $p_1(200) \leq 1 - x - y$. Thus $x \leq 0.2$ and $0.4 \leq 1 - x - y$

(d) (2.5 pt) p_1 is second order stochastically dominated by $\widetilde{p}$ if and only if $x \leq p_1(-100)$ and $2x + y \leq 2p_1(-100) + p_1(50)$. Thus $x \leq 0.2$ and $2x + y \leq 0.8$

2. **Pareto Efficiency with Quasi-Linear Preference**

Consider a pure exchange economy $\mathcal{E}^{pure} = \left(\{X_i, \ \succeq_i, \ e_i\}_{i=1,2} \right)$ with two goods and two consumers, where $X_i = \mathbb{R}^2_+$ and $e_1 + e_2 \gg 0$. Consumer i's preference is represented by a quasi-linear utility function $v_i\left(x_{i,1}\right) + x_{i,2}$, where v_i is a differentiable, increasing and strictly concave function.

(a) Define Pareto efficiency and show that an allocation $x = (x_1, x_2) \in \mathbb{R}^4_+$ is Pareto-efficient if and only if there exists $\overline{u}_2$ such that x solves the following problem:

$$\begin{aligned} &\max_{x_i \geq 0} v_1\left(x_{1,1}\right) + x_{1,2} \\ \text{s.t. } & v_2\left(x_{2,1}\right) + x_{2,2} \geq \overline{u}_2 \\ & \sum_{i=1}^{2} x_i \leq \sum_{i=1}^{2} e_i. \end{aligned}$$

(b) Write down the Kuhn-Tucker conditions for the problem in (a) and discuss briefly why they are necessary and sufficient for the optimal solutions (assume that both consumers consume a positive amount of good 1).

Note: For the next two questions (c) and (d), you can provide a graphical answer using the Edgeworth box . But it needs to be accompanied with a clear enough explanation.

(c) Show that, if x' and x'' are interior Pareto efficient allocations in $\mathbb{R}^4_{++}$, then $x'_{i,1} = x''_{i,1}$ for $i = 1, 2$. That is, each consumer consumes exactly the same amount of good 1 across all interior Pareto-efficient allocations.

(d) Suppose that $v_1\left(x_{1,1}\right) = \log x_{1,1}$, $v_2\left(x_{2,1}\right) = 2 \log x_{2,1}$, and $e_1 = e_2 = (1, 1)$. Find all Pareto efficient allocations and depict them in the Edgeworth box.

Answer for Q2

(a) (3 pt.)

The definition of PE is standard.

Suppose that x^* is Pareto-efficient. Then x^* must solve the above problem with $\overline{u}_2 = v_2\left(x^*_{2,1}\right) + x^*_{2,2}$. Otherwise, there exists a feasible allocation x' that guarantees at least $\overline{u}_2$ to consumer 2 and makes consumer 1 better off (i.e. $v_1\left({x'}_{1,1}\right) + x'_{1,2} > v_1\left(x^*_{1,1}\right) + x^*_{1,2}$). This is a contradiction. Conversely, suppose that x^* solves the above problem for some $\overline{u}_2$. If it is not Pareto-efficient, then there exists a feasible allocation x' that satisfies $v_i\left({x'}_{i,1}\right) + x'_{i,2} \geq v_i\left(x^*_{i,1}\right) + x^*_{i,2}$ for $i = 1, 2$ with at least one inequality being strict. We can find such x' for which the inequality is strict for consumer 1 (if the inequality is strict only for consumer 2, consumer 2's goods can be transferred to consumer 1). This is a contradiction because x^* is supposed to be an optimal solution for the above problem with $\overline{u}_2$.

(b) (2 pt.)

The Kuhn-Tucker conditions are

$$\begin{aligned}
\frac{dv_1\left(x_{1,1}\right)}{dx_{1,1}} - \lambda_1 &= 0,\ 1 - \lambda_2 \leq 0\ \ (= 0 \text{ if } x_{1,2} > 0) \\
\mu\frac{dv_2\left(x_{2,1}\right)}{dx_{2,1}} - \lambda_1 &= 0,\ \mu - \lambda_2 \leq 0\ \ (= 0 \text{ if } x_{2,2} > 0) \\
\mu\left(v_2\left(x_{2,1}\right) + x_{2,2} - \overline{u}_2\right) &= 0,\ \mu \geq 0,\ v_2\left(x_{2,1}\right) + x_{2,2} \geq \overline{u}_2, \\
\sum_{i=1}^{2} e_i &= \sum_{i=1}^{2} x_i \text{ (because } \lambda_\ell > 0\text{)}.
\end{aligned}$$

Since clearly μ must be positive and we can assume $\overline{u}_2 = v_2\left(x_{2,1}\right) + x_{2,2}$, it is OK to the following simpler conditions:

$$\begin{aligned}
\frac{dv_1\left(x_{1,1}\right)}{dx_{1,1}} - \lambda_1 &= 0,\ 1 - \lambda_2 \leq 0\ \ (= 0 \text{ if } x_{1,2} > 0) \\
\mu\frac{dv_2\left(x_{2,1}\right)}{dx_{2,1}} - \lambda_1 &= 0,\ \mu - \lambda_2 \leq 0\ \ (= 0 \text{ if } x_{2,2} > 0) \\
\sum_{i=1}^{2} e_i &= \sum_{i=1}^{2} x_i.
\end{aligned}$$

They are necessary and sufficient because (necessity:) the constraint set is a convex set (defined via upper contour sets of concave functions).and has an interior point (Slater condition, note that $x \gg 0$), and (sufficiency:) the objective function (utility function) is concave and the constraint set is a convex set.

(Note: The Edgeworth box can be used to answer (c) and (d) as long as clear enough explanation is offered).

(c) (2 pt) If $x_i \gg 0$ for $i = 1, 2$, then $\lambda_2 = 1$ hence $\mu = 1$. Thus $\frac{dv_1(x_{1,1})}{dx_{1,1}} = \frac{dv_2(x_{2,1})}{dx_{2,1}}$ (or this just follows from $MRS_1 = MRS_2$ which holds at any interior Pareto efficient allocation). Since v_i is strictly concave, there is at most one $(x_{1,1}, x_{2,1})$ that satisfies this and $\sum_{i=1}^{2} e_{i,1} = \sum_{i=1}^{2} x_{i,1}$.

(d) (3 pt) In this case, the conditions are

$$\begin{aligned}
\frac{1}{x_{1,1}} - \lambda_1 &= 0,\ 1 - \lambda_2 \leq 0\ \ (= 0 \text{ if } x_{1,2} > 0) \\
\frac{2\mu}{x_{2,1}} - \lambda_1 &= 0,\ \mu - \lambda_2 \leq 0\ \ (= 0 \text{ if } x_{2,2} > 0) \\
\sum_{i=1}^{2} x_i &= (2, 2).
\end{aligned}$$

There are three cases: (i) when $x_{i,2} > 0$ for $i = 1, 2$, then $x_{1,1} = \frac{2}{3}$ and $x_{2,1} = \frac{4}{3}$. So the solution is $(x_1, x_2) = \left(\left(\frac{2}{3}, m\right), \left(\frac{4}{3}, 2 - m\right)\right)$ with $m \in (0, 2)$ in this case, (ii) If $x_{1,2} = 0$ and $x_{2,2} = 2$, then $x_{1,1}$ and $x_{2,1}$ are determined by $\frac{1}{x_{1,1}} = \frac{2\mu}{x_{2,1}}$ with any $\mu \geqq 1$ and $x_{1,1} + x_{2,1} = 2$. So the solution is $(x_1, x_2) = \left(\left(\frac{2}{3} - \alpha, 0\right), \left(\frac{4}{3} + \alpha, 2\right)\right)$ with $\alpha \in \left[0, \frac{2}{3}\right)$ in this case, (iii) If $x_{1,2} = 2$ and $x_{2,2} = 0$, then $x_{1,1}$ and $x_{2,1}$ are determined by $\frac{1}{x_{1,1}} = \frac{2\mu}{x_{2,1}}$ with any $\mu \in (0, 1]$ and $x_{1,1} + x_{2,1} = 2$. So the solution is $(x_1, x_2) = \left(\left(\frac{2}{3} + \alpha, 2\right), \left(\frac{4}{3} - \alpha, 0\right)\right)$ with $\alpha \in \left[0, \frac{4}{3}\right)$ in this case. In addition to these, there are two extreme Pareto efficient allocations that are not captured by these KT conditions: $(x_1, x_2) = ((2, 2), (0, 0))$ and $(x_1, x_2) = ((0, 0), (2, 2))$.

Problem 3: Cournot Competition with Capacity Two firms produce an identical good for sale in a single market. Both firms have 0 fixed cost and 0 marginal cost. The market inverse demand function is

$$P = 1 - Q$$

Firms choose quantities and the market determines the price.

(i) Suppose first that capacity is *exogenous*: firm i has capacity $k_i \in [0, .5]$. If firms simultaneously choose actual quantities (subject to their capacity constraint), find the pure strategy NE of the one-shot game. [Suggestion: It may be easier to analyze case-by-case.]

(ii) Now suppose capacity is *endogenous*. Firms play a two-stage game. In the first stage, firms simultaneously choose capacities $k_i \in [0, .5]$; between stages, capacity choices are revealed; in the second stage firms simultaneously choose quantities (subject to their capacity constraint).

Find all the SGPE of the two-stage game.

Problem 3: Solution

If the firms choose quantities q_1, q_2 then the profit of Firm i is $\Pi_i(q_1, q_2) = q_i(1 - q_1 - q_2)$ and the derivative of profit is

$$\frac{\partial \Pi_i}{\partial q_i} = 1 - 2q_i - q_j$$

Note that this is positive provided that $q_1 \geq (1 - q_2)/2$. Hence the FOC is

$$1 - 2q_i - q_j = 0$$

The two FOC's have a unique simultaneous solution: $q_1 = q_2 = 1/3$.

(i) Consider four cases

(a) $k_1 \geq 1/3, k_2 \geq 1/3$: The FOC's have a simultaneous solution, so the unique NE is $q_1 = q_2 = 1/3$.

(b) $k_1 < 1/3, k_2 < 1/3$: Note that Π_i is increasing in q_i for both i so the unique solution is for both firms to choose quantity = capacity. Hence the unique NE is $q_1 = k_1, q_2 = k_2$.

(c) $k_1 < 1/3, k_2 \geq 1/3$: The FOC do not have a simultaneous solution so at least one firm must choose at its capacity constraint. If firm 1 is NOT choosing at its capacity constraint firm 2 must be choosing at its capacity constraint so $q_2 = k_2$ and the optimal choice for firm 1 is $q_1 = (1 - k_2)/2$ and the optimum for firm 2 is

$$q_2 = (1 - q_1)/2 = [1 - (1 - k_2)/2]/2 = 1/4 + k_2/4$$

Because $k_2 > 1/3$, the optimal choice for firm 2 is feasible so choosing at its capacity constraint is not optimal; this is a contradiction. Hence we conclude that firm 1 MUST be choosing at its capacity constraint: $q_1 = k_1$.

Hence firm 2's optimal choice is $q_2 = (1 - k_1)/2$ if this is feasible, so we have two possibilities:

(1) $k_2 > (1 - k_1)/2$ in which case the unique NE is $q_1 = k_1, q_2 = (1 - k_1)/2$.

(2) $k_2 \leq (1 - k_1)/2$ in which case the unique NE is $q_1 = k_1, q_2 = k_2$.

(d) $k_1 \geq 1/3, k_2 < 1/3$ This is the same as (c) except with the roles of firms 1, 2 reversed.

(ii) CLAIM: In every SGPE of the two-stage game, we must have $k_1 \geq 1/3, k_2 \geq 1/3$. To see this, notice that in all the other cases at least one firm sets capacity below 1/3; say $k_1 < 1/3$. In a SGPE, firms are playing NE in the second stage, so we must be in either case (b) or case (c) of part (i). In either case compute the profit of firm 1 and note that it is strictly increasing in k_1; hence firm 1 would have a profitable deviation in the first stage, a contradiction.

Hence in every SGPE of the two-stage game, play in the first stage must be $k_1 \geq 1/3, k_2 \geq 1/3$ and by case (a) of part (i) play in the second stage must be $q_1 = 1/3, q_2 = 1/3$.

CONCLUSION: The SGPE are precisely these:

- **First Stage** $k_1 \geq 1/3, k_2 \geq 1/3$.
- **Second Stage** IF $k_1 \geq 1/3, k_2 \geq 1/3$ then $q_1 = 1/3, q_2 = 1/3$. OTHERWISE play is according to the specifications of cases (b), (c), (d) of part (i).

Problem 4: Repeated Differentiated Commodities Two firms produce different goods for sale in a single market. Each firm's production imposes a negative externality on the other: if the firms produce quantities q_1, q_2 then their profits will be

$$\begin{aligned} \Pi_1(q_1, q_2) &= (120 - q_2)q_1 - q_1^2 \\ \Pi_2(q_1, q_2) &= (120 - q_1)q_2 - q_2^2 \end{aligned}$$

(i) Suppose first that the firms interact only once. Find the (pure strategy)Nash equilibrium of the game and the (symmetric) Pareto optimum.

(ii) Now suppose the firms interact infinitely often and discount future profits with the discount factor $\delta < 1$.

  (a) For what values of δ (if any) is there a SGPE in which firms play the (symmetric) Pareto optimum in each period in which there has been no deviation and firms punish any deviation by permanent reversion to the one shot Nash equilibrium?

  (b) For what values of δ (if any) is there a SGPE in which firms play the (symmetric) Pareto optimum in each period in which there has been no deviation and punish *any* deviation by playing *one round* of min-max against the deviator? (After punishment is complete and there has been no deviation from punishment play returns to the (symmetric) Pareto optimum.)

Problem 4: Solution

(i) At the symmetric Pareto Optimum firms choose $q_1 = q_2 = x$ to maximize $2[(120 - x)x - x^2]$. The FOC is

$$0 = 2[120 - x - x - 2x] = 240 - 8x$$

So the symmetric PO is $q_1 = q_2 = x = 30$ and each firm's profit is 1800.

To solve for the one-shot NE, write the FOC:

$$\begin{aligned} 120 - q^2 - 2q^1 &= 0 \\ 120 - q^1 - 2q^2 &= 0 \end{aligned}$$

so the one-shot NE is $q_1 = q_2 = 40$ and each firm's profit is 1600.

(ii)(a) While firms are playing the symmetric PO then each firm earns 1800 per period. An optimal one-period deviation by either firm is to choose y to maximize $(120 - 30)y - y^2$, so the optimal deviation is $y = 45$. This gives the deviating firm one-period profit of $(120 - 30)(45) - (45)^2 = 2025$. Hence the firm gains $2025 - 1800 = 225$ once but (since play reverts to the one-shot NE) it loses $1800 - 1600 = 200$ in each subsequent period. Hence deviation is NOT profitable if

$$225 - 200\delta/(1 - \delta) \leq 0$$

Hence deviation is not profitable and this is a SGPE if $\delta \geq 225/425 = 9/17$.

(ii)(b) While firms are playing the symmetric PO then each firm earns 1800 per period. An optimal one-period deviation by firm 1 (say) is to choose $q_1 = y = 45$ so firm 1 gains $2025 - 1800 = 225$ once. After the deviation, firm 1 will be punished in the next period by the minmax play which is $q_1 = 0, q_2 = 120$ so it will lose 1800 in the next period, after which play reverts to PO again, so deviation by firm 1 is NOT profitable if

$$225 - 1800\delta \leq 0$$

Hence deviation is not profitable if $\delta \geq 225/1800 = 1/8$.

However this is *not enough* to guarantee that this strategy is a SGPE because we have to check that the punishment is part of a SGPE. In (ii)(a) this is automatic because playing a one-shot NE in every period is always a SGPE; but playing minmax in every period is not a SGPE. Here we have to check that firm 2 is willing to punish. If firm 2 does punish, it gets 0 this period and (because play reverts to PO afterward) 1800 in each succeeding period. A maximal deviation by firm 2 is to choose $q_2 = z$ to maximize $(120-0)z - z^2$ to the maximal deviation is $z = 60$. If firm 2 deviates its one-period profit will be $(120 - 0)(60) - (60)^2 = 3600$. But then firm 2 will be minmaxed for one period – in which it will get 0 – and then play will return to PO ever after. Hence firm 2 will gain 3600 in the current period and lose 1800 in only one subsequent period – so firm 2 will always gain by deviating from this punishment, no matter what δ is. Hence there is no δ for which this is a SGPE. [Note: In order to construct a SGPE it would be necessary for the punishment to last more than one period; this would provide the incentive firm 2 to carry out the punishment.]