Fall 2015

1. **Representative Producer**

Consider n producers with production possibility set $Y_j \subset \mathbb{R}^L$, $j = 1, ..., n$. Each Y_j is nonempty and closed. Let $\pi_j(p) = \sup_{y_j \in Y_j} p \cdot y_j$ be the maximum possible profit and $y_j(p)$ is the set of optimal productions (which may be empty) given price vector $p \in \mathbb{R}^L$. Also consider a hypothetical producer who has an access to all these production possibility sets in the sense that he is equipped with a total production possibility set $Y = \sum_{j=1}^{n} Y_j$. Let $\pi(p)$ and $y(p)$ be the profit function and the optimal production correspondence with respect to Y respectively. Answer the following questions.

(a) Suppose that each Y_j is convex. Does this mean that Y is convex? If so, prove it. If not, find a counterexample. Also do the converse. Suppose that Y is convex. Does this mean that each Y_j is convex?

(b) Suppose that Y is bounded above, i.e. there exists $\overline{y} \in \mathbb{R}^L$ such that $\overline{y} \geq y$ for any $y \in Y$. Show that $y(p)$ is nonempty given any $p \in \mathbb{R}^L_{++}$.

(c) Show that $\pi(p) = \sum_{j=1}^{n} \pi_j(p)$. Also show that $y(p) = \sum_{j=1}^{n} y_j(p)$ when $y(p)$ is nonempty (convexity/boundedness is not assumed for this question).

Answer for Q1

(a-i) (1.5 pts.)

Yes. Take any $y_k \in Y, k = 1, .., K$. For each k, there exists $y_{j,k} \in Y_j$ $j = 1, .., n$ such that $y_k = \sum_{j=1}^{n} y_{j,k}$. So, for any weights $\lambda_k \geq 0, k = 1, .., K$, such that $\sum_{k=1}^{K} \lambda_k = 1$, we have

$$\sum_{k=1}^{K} \lambda_k y_k = \sum_{k=1}^{K} \lambda_k \sum_{j=1}^{n} y_{j,k} = \sum_{j=1}^{n} \sum_{k=1}^{K} \lambda_k y_{j,k}.$$

Since Y_j is convex, $\sum_{k=1}^{K} \lambda_k y_{j,k} \in Y_j$ for each j. Hence $\sum_{k=1}^{K} \lambda_k y_k \in Y$.

(a-ii) (1.5 pts)

No. For example, if $Y_1 = \mathbb{R}^L$, then $Y = \sum_{j=1}^{n} Y_j$ is $\mathbb{R}^L$ whatever Y_j is.

(b) (3 pts.)

Pick any $\underline{y} \in Y$. We can write down the profit maximization problem of the representative producer given any p as follows.

$$\sup_{y \in Y} p \cdot y, \text{ s.t. } p \cdot y \geq p \cdot \underline{y}.$$

- The objective function is clearly continuous in y.
- The constraint set is compact for any $p \gg 0$: It's bounded above by assumption. Suppose that it is not bounded below and it is possible to pick an arbitrary small y_ℓ for some good ℓ. This implies that $y_{\ell'}$ must be arbitrary large for some good ℓ' to satisfy $p \cdot y \geq p \cdot \underline{y}$ because $p \gg 0$. This is a contradiction. So it must be bounded below as well. Hence the constraint set is compact.

Therefore the solution set for this problem, which is $y(p)$, is not empty for any $p \in \mathbb{R}^L_{++}$.

(c) (4 pts.)

Take any $y \in Y$, then $y = \sum_{j=1}^{n} y_j$ for some $y_j \in Y_j$ for $j = 1, .., n$. So $p \cdot y = \sum_{j=1}^{n} p \cdot y_j \leq \sum_{j=1}^{n} \pi_j(p)$, hence $\pi(p) \leq \sum_{j=1}^{n} \pi_j(p)$. Conversely, for any

$y_j \in Y_j, j = 1, ..., n,\ \sum_{j=1}^{n} p \cdot y_j = p \cdot \sum_{j=1}^{n} y_j \leq \pi(p)$. Hence $\sum_{j=1}^{n} \pi_j(p) \leq \pi(p)$. Therefore $\pi(p) = \sum_{j=1}^{n} \pi_j(p)$ (which may be ∞).

$y(p) = \sum_{j=1}^{n} y_j(p)$ can be shown as follows.

- $y(p) \subset \sum_{j=1}^{n} y_j(p)$: Take any $y' \in y(p)$. Then $y' = \sum_{j=1}^{n} y'_j$ for some $y'_j \in Y_j$ for $j = 1, .., , n$. Then y'_j must be profit maximizing for producer j. Hence $y'_j \in y_j(p)$. Therefore $y' \in \sum_{j=1}^{n} y_j(p)$.

- $y(p) \supset \sum_{j=1}^{n} y_j(p)$: Take any $y'_j \in y_j(p), j = 1, ..., n$. Then $p \cdot \sum_{j=1}^{n} y'_j = \sum_{j=1}^{n} \pi_j(p) = \pi(p)$ (by the above result). Hence $\sum_{j=1}^{n} y'_j \in y(p)$.

2. **Revealed Preference**

Suppose that you collect a finite data of price-consumption vector pairs $D = \{(p^t, x^t) \in \mathbb{R}^L_{++} \times \mathbb{R}^L_{+}, t = 1, ..., T\}$ by asking the same consumer which consumption bundle he would chose if the prices were $p^t, t = 1, ..., T$. Assume that his choice would be based on his preference $\succeq$ and his answer is honest.

D satisfies WARP if x^t is revealed preferred to x^s (i.e. $p^t \cdot x^t \geq p^t \cdot x^s$ and $x^t \neq x^s$), then x^s is not revealed preferred to x^t for any s, t. D satisfies GARP if x^t is indirectly revealed preferred to x_s (i.e. x^t is revealed preferred to x^{t_1}, x^{t_1} is revealed preferred to x^{t_2}, ... and x^{t_K} is revealed preferred to x^s), then x^s is not strictly revealed preferred to x^t (i.e. $p^s \cdot x^t \geq p^s \cdot x^s$) for any s, t. Answer the following questions.

(a) Suppose that this consumer maximizes the following utility function $u(x) = \min \left\{ \sum_{\ell=1}^{L} \alpha_\ell x_\ell, \sum_{\ell=1}^{L} \alpha'_\ell x_\ell \right\}$ for some parameters $\alpha, \alpha' \in \mathbb{R}^L_{++}$. Discuss whether D must satisfy WARP and/or GARP or not in this case. If it may not satisfy either WARP or GARP, find such an example of D.

(b) Suppose that this consumer chooses optimally with respect to the following Lexicographic preference: $x \succeq x'$ if and only $x_\ell \geq x'_\ell$ for $\ell = 1, ..., L'$ and $x_{L'+1} > x'_{L'+1}$ for some $L' \in \{1, ..., L\}$. Discuss whether D must satisfy WARP and/or GARP or not in this case. If it may not satisfy either WARP or GARP, find such an example of D.

(c) Discuss briefly why no lexicographic preference would be needed to rationalize D even if this consumer has the lexicographic preference as described in (b).

Answer for Q2

(a) (4 pts.)

- (2 pts.) Note that the preference represented by this utility function (which is a kind of generalized Leontief function) is locally nonsatiated. D must satisfy GARP when the underlying preference is locally nonsatiated for the following reason. If x^t is revealed preferred to $x^{t'}$, then $x^t \succeq x^{t'}$ by utility maximization. So, if x^t is indirectly revealed preferred to x^s, then $x^t \succeq x^s$ must hold. Now suppose that x^s is strictly revealed preferred to x^t. Then there exists x' nearby x^t such that $p^s \cdot x' < p^s \cdot x^s$ and $x' \succ x^t \succeq x^s$ by local nonsatiation. This is a contradiction to x^s being the utility maximizing. Hence x^s must not be strictly revealed preferred to x^t.

- (2 pts.) D may not satisfy WARP. For example, suppose that $L = 2$ and $u(x) = \min\{2x_1 + x_2, x_1 + 2x_2\}$. Then both $(p^t, x^t) = ((2,1), (1,1))$ and $(p^s, x^s) = ((2,1), (0.5, 2))$ can be a utility maximizing consumption, but WARP is violated as $p^t \cdot x^t = p^t \cdot x^s = p^s \cdot x^t = p^s \cdot x^s = 3$ and $x^t \neq x^s$.

(b) (3 pts.)

- (1 pt) The lexicographic preference is locally nonsatiated. So D must satisfy GARP as shown above.

- (2 pts.) It satisfies WARP as well. Suppose that x^t is revealed preferred to x^s. Note that this consumer is never indifferent between any two distinct consumption bundle x^t and x^s. Hence $x^t \succ x^s$ must hold. But then x^s cannot be strictly revealed preferred to x^t. Thus WARP holds.

(c) (3 pts.) GARP is satisfied given the above lexicographic preference. But, by Afriat's theorem, D can be rationalized by some continuous, concave and strictly increasing utility function whenever GARP is satisfied. So we would not need any lexicographic preference to rationalize/justify D.

#3) (Bayesian Games) Rose and Carl; live in NY City and use the subways to play an ongoing game of tag. Rose takes the 1st Avenue subway and gets off at either the Market Street station or the South Street station. Carl will go to either Market Street or South Street; he can go to either station – but not both. If Carl goes to the wrong station, he will *never* tag Rose; if he goes to the right station *and* the station *is not crowded* he will tag rose for sure but if he goes to the right station and the station *is crowded* he will only tag Rose 50% of the time.

Rose and Carl agree that not tagging is worth +10 to Rose and −10 to Carl, and that tagging is worth −10 to Rose and +10 to Carl. Riding the subway is unpleasant; the extra cost to each of them to go to Market Street rather than to South Street is −2, no matter what else happens.

It is common knowledge that each station will be busy 50% of the time and that the conditions at the two stations are independent. Carl cannot see the conditions before he makes his decision where to go. Rose can see the condition at the South Street station (which is the first stop) but not the Market Street station *before* she decides where to get off.

- Model this situation as a Bayesian Game and identify all the elements.
- Find all the Bayesian Nash Equilibria. (You may find it easier to solve in behavioral strategies.)

Solution

(a) Only Rose has private information (whether Market Street station is crowded) so she has two types: call them C, NC (crowded, not crowded). Carl has only one type. So a pure strategy for Carl is a choice of where to go: M, S (two pure strategies) but a pure strategy for Rose is a function from her types to her choices: $\{C, NC\} \to \{M, S\}$ (four pure strategies).

Payoff matrices depend on Christine's type – whether or not Market Street is crowded. Payoffs are computed according to the probability of being tagged and the extra cost of riding the subway from Market Street to South Street. Ralph's actions are shown as Rows and his payoffs come first. Actions M = Market Street, S = South Street.

Rose type = B; ex ante probability = .5

	M	S
M	0,0	+10,-12
S	+8,-10	-2,-2

Rose type = NB; ex ante probability = .5

	M	S
M	-10,+10	+10,-12
S	+8,-10	-2,-2

(b) First show there is NO BNE in pure strategies.

- If Carl always goes to Market Street then Rose's best reply is to always go to South Street; but then Carl is not optimizing. If Carl always goes to South Street then Rose's best reply is to always go to Market Street; but then Carl is not optimizing. Hence there is no BNE in which Carl uses a pure strategy.

- If Rose always goes to Market Street then Carl's best reply is to always go to Market Street; if Rose always goes to South Street then Carl's best reply is to always go to South Street. In either case, Carl's best response cannot be a mixed strategy.

- If Rose always goes to Market Street when her type is B and to South Street when her type is NB then Carl's best response is to go to Market Street (because the ex ante probability of Rose's type being B is .5) so Carl's best response cannot be a mixed strategy.

- If Rose always goes to South Street when her type is B and to Market Street when her type is NB then Carl's best response is again to go to Market Street (because the ex ante probability of Rose's type being B is .5) so Carl's best response cannot be a mixed strategy.

Hence Carl must use a mixed strategy – say Carl plays $cM + (1-c)S$ where $0 < c < 1$ – and that at least one type of Rose must use a mixed strategy.

But if a one type of Rose uses a mixed strategy then that type is indifferent between its two pure strategies. Check that it is impossible for *both* types of Rose to be indifferent so only one type mixes.

If Rose of type B mixes $bM + (1-b)S$

- Solve and find $c = .4$.
- Type NB strictly prefers to play S.
- In order to make Carl indifferent we must have $b = .9$

This BNE equilibrium is

$$R - B : .9M + .1S, R - NB : S; C : .4M + .6S$$

If Rose of type NB mixes $b'M + (1-b')S$ this does not lead to a BNE.

Conclusion: unique BNE

- Rose of type B Plays $.9M + .1S$
- Rose of type NB plays S
- Carl plays $.4M + .6S$

#4 (Repeated Games)

	L	R
U	2,2	8,0
D	0,8	12,1

a) Find all the Nash equilibria in pure strategies.

The following two questions concern the infinitely repeated game $\Gamma^{\infty}(\delta)$: the game Γ is played in each period and both players discount future payoffs using the common discount factor δ. [The discount factor δ is a parameter.]

b) If $\delta \leq 1/4$, show that there does NOT exist a subgame perfect equilibrium strategy profile σ of $\Gamma^{\infty}(1/4)$ and a history h such that $\sigma(h) = UR$.

c) If $\rho \geq 3/4$ find a subgame perfect equilibrium strategy profile σ in which play along the equilibrium path alternates between UR and DL, and verify that your profile is in fact subgame perfect.

a) The unique NE in pure strategies is UL.

b) Suppose that σ is a SGPE and that $\sigma(h) = UR$. If ROW obeys according to σ then his current payoff will be 8 and the highest possible continuation payoff is 12 so his average payoff will be at most $8(1-\delta) + 12\delta$. If ROW deviates from σ and plays D his current payoff will be 12 and the lowest possible continuation payoff will be 2 so his average payoff will be at least $12(1-\delta)+2\delta$. Because $\delta \leq 1/4$, deviating is better than obeying (the actual cutoff for obeying to be as good as deviating is $\delta = 2/7 > 1/4$) so σ is not a SGPE.

c) Define σ as follows

- If there has never been a deviation and the length of the history h is even then $\sigma(h) = UR$.
- If there has never been a deviation and the length of the history h is odd then $\sigma(h) = DL$.
- If there has ever been a deviation then $\sigma(h) = UL$.

To see that this is a SGPE we need to show three things

- If there has never been a deviation then ROW does not wish to deviate.

  Suppose current play is UR. If ROW deviates then he gets 12 this period and 2 in every succeeding period so his long run average is $12(1-\delta)+2\delta$. If ROW obeys then he gets 8 this period then 0 next period then 8 ...so his long run average is $8/(1+\delta)$. Because $\delta \geq 3/4$ obeying is at least as good as deviating. (The actual cutoff is $\delta = (2+\sqrt{164})/20$.)

  Suppose current play is DL. If ROW deviates he gets 2 this period and 2 in every succeeding period so his long run average is 2. If ROW obeys then he gets 0 this period then 8, then 0, ...so his long run average is $8\delta/(1+\delta)$. Because $\delta \geq 3/4$ obeying is at least as good as deviating. (The actual cutoff is $\delta = 1/3$.

- If there has never been a deviation then COL does not wish to deviate.

  This is the same calculation as for ROW.

- If there has been a deviation then neither player wishes to deviate.

  This is obvious since after a deviation the prescribed play is always the NE so neither player gains by deviating.