Fall 2014

1. **First Welfare Theorem**

   Let $\mathcal{E}^{pure} = (\{X_i, \succeq_i, e_i\}_{i \in I})$ be the standard pure exchange economy with free disposal, where $X_i = \mathbb{R}^L_+$ and $\succeq_i$ is locally nonsatiated for every $i \in I$. Answer the following questions.

   (a) Define Walrasian equilibrium and Pareto efficient allocation in this economy.

   (b) Prove that every Walrasian equilibrium allocation is Pareto efficient.

   (c) Suppose that $I = \{1, 2, 3\}$. Suppose that consumer 1 and consumer 2 decide to trade exclusively with each other, effectively excluding consumer 3 from any exchange. Consumer 1 and 2 negotiate to come up with a pair of consumption vectors $x'_1, x'_2 \in \mathbb{R}^L_+$ such that $x'_1 + x'_2 \leq e_1 + e_2$. Of course consumer 3 just consumes her endowment e_3 (or a part of it). Let $(x^*, p^*) \in \mathbb{R}^{3L}_+ \times \mathbb{R}^L_+$ be any Walrasian equilibrium that would have realized if every consumer can participate in the market. Clearly consumer 3 is always (weakly) worse off by consuming e_3 rather than x^*_3. But is it possible that consumer 1 and 2 are better off negotiating with each other, i.e. $x'_i \succeq_i x^*_i$ for $i = 1, 2$ and $x'_i \succ_i x^*_i$ for $i = 1$ or 2? If so, find such an example. If not, explain why.

Answer for Q1:

(a) Let $r = \sum_i e_i \gg 0$ be the total resource of the economy. $(x^*, p^*) \in X \times \mathbb{R}^L_+$ is a *Walrasian equilibrium* in $\mathcal{E}^{pure}$ (**Note:** Free disposal implies nonnegative prices) if

(i) $x_i^* \succeq_i x_i$ for all $x_i \in B_i(p^*, p^* \cdot e_i)$ and $x_i^* \in B_i(p^*, p^* \cdot e_i)$ for all $i \in I$

(ii) $\sum_i x_i^* \leq r$ (and $\sum_i x_{i,\ell}^* = r_\ell$ if $p_\ell^* > 0$).

Let $A = \{x \in X : \sum_i x_i \leq r\}$ be the set of feasible allocations in $\mathcal{E}^{pure}$. An allocation $x \in X$ is *Pareto efficient* if $x \in A$ and there does not exist any $x' \in A$ such that $x_i' \succeq_i x_i$ for all $i \in I$ and $x_i' \succ_i x_i$ for some $i \in I$.

(b) Suppose that $(x^*, p^*) \in X \times \mathbb{R}^L_+$ is a Walrasian equilibrium in $\mathcal{E}^{pure}$ and that x^* is not Pareto efficient. Since $x^* \in A$ by (ii) of the definition of WE, there exist $x' \in A$ and $k \in I$ such that $x_i' \succeq_i x_i^*$ for all $i \in I$ and $x_k' \succ_k x_k^*$. If $p^* \cdot e_i > p^* \cdot x_i'$, then we can find x_i'' such that $x_i'' \succ_i x_i' \, (\succeq_i x_i^*)$ and $p^* \cdot e_i > p^* \cdot x_i''$ by local nonsatiation. This is a contradiction. Hence, $p^* \cdot x_i' \geq p^* \cdot e_i$, for every $i \in I$. Utility maximization of consumer k implies $p^* \cdot x_k' > p^* \cdot e_k$.

Summing up these inequalities over $i \in I$, we would obtain $p^* \cdot \sum_i x_i' > p^* \cdot \sum_i e_i$. On the other hand, since $\sum_i x_i' \leq \sum_i e_i$.and $p^* \in \mathbb{R}^L_+$, we have $p^* \cdot \sum_i x_i' \leq p^* \cdot \sum_i e_i$. This is a contradiction. Therefore, x^* must be Pareto efficient when $(x^*, p^*) \in X \times \mathbb{R}^L_+$ is a Walrasian equilibrium.

(c) Suppose that $x_i' \succeq_i x_i^*$ for $i = 1, 2$ and $x_i' \succ_i x_i^*$ for $i = 1$ or 2. Without loss of generality, assume that consumer 1 strictly prefers x_1' to x_1^*. As in (b), we can show $p^* \cdot x_1' > p^* \cdot e_1$ and $p^* \cdot x_2' \geqq p^* \cdot e_2$. Hence $p^* \cdot \sum_{i=1,2} x_i' > p^* \cdot \sum_{i=1,2} e_i$. On the other hand, $\sum_{i=1,2} x_i' \leqq \sum_{i=1,2} e_i$ implies $p^* \cdot \sum_{i=1,2} x_i' \leqq p^* \cdot \sum_{i=1,2} e_i$, which is a contradiction. Hence such exclusive arrangement does not make both consumer 1 and 2 better off.

2. **Pareto Efficiency and Externality**

We consider a pure exchange economy $\mathcal{E}^{ext}$ with consumption externality, where consumer 1's utility is directly affected by other consumers' consumptions. Consumer 1's preference can be represented by a continuous utility function $u_1(x) = f(x_1) - \sum_{i \neq 1} g_i(x_i)$, where f is increasing ($x_1'' \gg x_1' \Rightarrow f(x_1'') > f(x_1')$) and concave and g_i is increasing and convex. The preference of consumer $i \, (\neq 1)$ is represented by a usual utility function $u_i(x_i)$, which is increasing and concave.

(a) Let $U = \{u \in \mathbb{R}^I_+ : \exists x \text{ feasible}, u \leq u(x)\}$ be the utility possibility set. Show that U is closed and convex.

(b) Show that a feasible allocation $x \in X$ in this economy $\mathcal{E}^{ext}$ is Pareto efficient if and only if it maximizes the weighted sum of utilities with respect to some weight vector $(a_1, ..., a_I) \in \mathbb{R}^I_{++}$.

(c) Consider the following two-good pure exchange economy with consumption externality: $I = \{1, 2\}$, $u_1(x) = \ln x_{1,1} + \ln x_{1,2} - x_{2,1}$, $u_2(x_2) = \ln x_{2,1} + \ln x_{2,2}$, and $e_1 = e_2 = (1/2, 1/2)$. The definition of Walrasian equilibrium $(x^*, p^*) \in \mathbb{R}^4_+ \times \mathbb{R}^2_+$ is the same as usual, except that x_1^* solves $\max_{x_1 \in \mathbb{R}^2_+} u_1(x_1, x_2^*)$, s.t. $p^* \cdot x_1 \leqq p^* \cdot e_1$ given x_2^*. Characterize the set of Pareto efficient allocations in $\mathbb{R}^4_{++}$ and show that every Walrasian equilibrium allocation in $\mathbb{R}^4_{++}$ is Pareto inefficient.

Answer for Q2:

(a)

Closedness: Take any convergent sequence u_t in U that converges to u^*. Let $x_t \in A$ be the sequence of associated feasible allocations such that $u_t \leqq u(x_t)$. Since the set of feasible allocations is compact, there exists a subsequence along which x_t converges to some $x^* \in A$. Then, by continuity of u, $u^* \leqq u(x^*)$. Hence $u^* \in U$.

Convexity: Take any u' and u'' in U. Then there exist $x', x'' \in A$ such that $u' \leqq u(x')$ and $u'' \leqq u(x'')$. For any $\lambda \in [0,1]$, $\lambda u' + (1-\lambda)u'' \leqq \lambda u(x') + (1-\lambda)u(x'') \leqq u(\lambda x' + (1-\lambda)x'')$ by convexity of u. Since A is convex, $\lambda x' + (1-\lambda)x''$ is feasible. Hence $\lambda u' + (1-\lambda)u'' \in U$ for any $\lambda \in [0,1]$.

(b) (Note: This result holds for an interior Pareto efficient allocation.) Suppose that $x^* \in \mathbb{R}^{I\times L}_{++}$ solves $\max_{x\in A} \sum_{i\in I} a_i u_i(x)$ for some $(a_1, ..., a_I) \in \mathbb{R}^I_{++}$. Suppose that it is not Pareto efficient. Then there exists $x' \in A$ such that $u_i(x') \geqq u_i(x^*)$ for all $i \in I$ and $u_i(x') > u_i(x^*)$ for some $i \in I$. Then $\sum_{i\in I} a_i u_i(x') > \sum_{i\in I} a_i u_i(x^*)$, a contradiction. Hence $x^* \in \mathbb{R}^{I\times L}_{++}$ must be Pareto efficient.

Conversely, suppose that $x^* \in \mathbb{R}^{I\times L}_{++}$ is Pareto efficient. U is closed and convex by (a). Note that $u(x^*)$ is clearly a boundary point of U. So, by the supporting hyperplane theorem, there exists $(a_1, ..., a_I)\,(\neq \mathbf{0}) \in \mathbb{R}^I_+$ such that $\sum_{i\in I} a_i u_i(x^*) \geqq \sum_{i\in I} a_i u_i$ for any $u \in U$, which implies that $\sum_{i\in I} a_i u_i(x^*) \geqq \sum_{i\in I} a_i u_i(x)$ for any $x \in A$.

Finally we show that each a_i is strictly positive. Suppose $a_j = 0$ for some j. If $j = 1$, pick a small $\varepsilon \in \mathbb{R}^L_{++}$ and let $x'_1 = x^*_1 - \varepsilon$, $x'_i = x^*_i + \frac{1}{n-1}\varepsilon$ for $i \neq 1$. x' is clearly feasible if ε is small enough. If $j \neq 1$ and $a_1 > 0$, then pick a small $\varepsilon \in \mathbb{R}^L_{++}$ and let $x'_1 = x^*_1 + \varepsilon$, $x'_j = x^*_j - \varepsilon$, $x'_i = x^*_i$ for $i \neq 1, j$. x' is clearly feasible if ε is small enough. In the first case, $u_i(x'_i) > u_i(x^*_i)$ for all $i \neq 1$. In the second case, $u_1(x') > u_1(x^*)$ and $u_i(x'_i) = u_i(x^*_i)$ for all $i \neq 1, j$. In either case, $\sum_{i\in I} a_i u_i(x') > \sum_{i\in I} a_i u_i(x^*)$, which is a contradiction. Hence $a \in \mathbb{R}^I_{++}$.

(c) By (b), $x^* \in \mathbb{R}^{I\times L}_{++}$ is Pareto efficient if and only if it solves $\max_{x\in A} \sum_{i\in I} a_i u_i(x)$ for some $(a_1, ..., a_I) \in \mathbb{R}^I_{++}$. Hence the following conditions characterize the set of Pareto efficient allocations in $\mathbb{R}^4_{++}$.

$$
\begin{aligned}
\exists\,(a_1, a_2) &\in \mathbb{R}^2_{++} \text{ and } \exists\,(\lambda_1, \lambda_2) \in \mathbb{R}^2_{+} \text{ such that} \\
a_1 \frac{1}{x_{1,1}} &= \lambda_1,\ a_1 \frac{1}{x_{1,2}} = \lambda_2 \\
-a_1 + a_2 \frac{1}{x_{2,1}} &= \lambda_1,\ a_2 \frac{1}{x_{2,2}} = \lambda_2 \\
x_{1,1} + x_{2,1} &= 1,\ x_{1,2} + x_{2,2} = 1.
\end{aligned}
$$

Eliminating a and λ, we have

$$
\begin{aligned}
\frac{x_{1,2}}{x_{1,1}} &= -x_{1,2} + \frac{x_{2,2}}{x_{2,1}} \\
x_{1,1} + x_{2,1} &= 1,\ x_{1,2} + x_{2,2} = 1.
\end{aligned}
$$

The following conditions are necessary and sufficient for Walrasian equilibrium.

$$
\begin{aligned}
\exists\,(p_1^*, p_2^*) &\in \mathbb{R}^2_{+} \text{ and } \exists\,(\lambda_1, \lambda_2) \in \mathbb{R}^2_{+} \text{ such that} \\
\frac{1}{x_{1,1}} &= \lambda_1 p_1^*,\ \frac{1}{x_{1,2}} = \lambda_1 p_2^* \\
\frac{1}{x_{2,1}} &= \lambda_2 p_1^*,\ \frac{1}{x_{2,2}} = \lambda_2 p_2^* \\
x_{1,1} + x_{2,1} &= 1,\ x_{1,2} + x_{2,2} = 1.
\end{aligned}
$$

Eliminating p^* and λ, we have

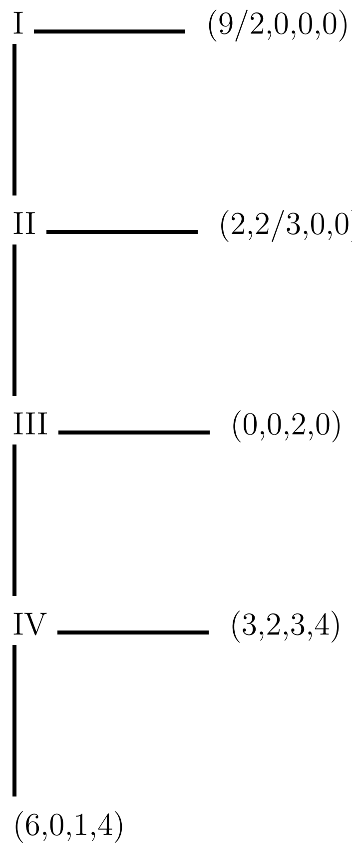
$$
\begin{aligned}
\frac{x_{1,2}}{x_{1,1}} &= \frac{x_{2,2}}{x_{2,1}} \\
x_{1,1} + x_{2,1} &= 1,\ x_{1,2} + x_{2,2} = 1.
\end{aligned}
$$

Clearly no $x \in \mathbb{R}^4_{++}$ would satisfy the conditions for PE and the conditions for WE simultaneously. So no Walrasian equilibrium allocation in $\mathbb{R}^4_{++}$ is Pareto efficient.

1. The diagram below shows a 4-player game.

   (a) Find *all* the subgame perfect equilibria in *pure strategies*.

   (b) Find *all* the subgame perfect equilibria (if any) in which player I plays a completely mixed strategy.



Solution

(a) Solve by backwards induction. Note that IV is indifferent to A (across) or D (down)

- If IV plays D then III strictly prefers to play A (because 2 ¿ 1)
  If III plays A then II strictly prefers to play A (because 2/3 ¿ 0)
  If II plays A then I strictly prefers to play A (because 9/2 ¿ 2)
  Hence one SGPE is **(A, A, A, D)**
- If IV plays A then III strictly prefers to play D (because 3 ¿ 2)
  If III plays D then II strictly prefers to play D (because 2 ¿ 2/3)
  If II plays D then I strictly prefers to play A (because 9/2 ¿ 3)
  Hence one SGPE is **(A, D, D, A)**

(b) If I plays a strictly mixed strategy then s/he must be indifferent between A, D; hence the expected payoff to I from playing D must be exactly 9/2. In order for the expected payoff to I from playing D to be 9/2, it must be the case that IV plays D with probability $p \geq .5$. If $p > .5$ then III strictly prefers to play A in which case the payoff to I will be at most 2, hence we must have $p = .5$. Now III is indifferent between A, D. If III plays A with strictly positive probability then the expected payoff to I is less than 9/2; hence III plays D. Now II strictly prefers to play D. Given all of this, I is indifferent so her/his play is arbitrary. Hence the set of SGPE in which I plays a strictly mixed strategy is $\{(\mathbf{qA + (1-q)D, D, D, .5A + .5D}) : \mathbf{0 < q < 1}\}$.

2. The government must decide whether to build a project that is of potential value to two firms. The cost of the project is c; the value to firm 1 is either 1 or 0, the value to firm 2 is either 2 or 0; in each case the probability of a positive value is p (where $0 < p < 1$) and the probabilities are independent. Whenever the government decides to build the project it will divide the cost c between the firms but will *never make a profit or provide a subsidy.*

   The government wants to use a socially efficient mechanism: that is, a mechanism that causes the project to be built if and only the cost is less than the total value to the firms. (To avoid complications we will ignore cases where the cost might be exactly equal to the total value to the firms.)

   (a) If $2 < c < 3$, find a socially efficient mechanism that is incentive compatible and (interim) individually rational for the firms. (That is, the firms are willing to participate in the mechanism after they know their true values.)

   (b) If $1 < c < 2$ find a socially efficient mechanism that is incentive compatible and (interim) individually rational for the firms.

   (c) If $0 < c < 1$ find the region in cost c and probability p space for which there is a a socially efficient mechanism that is incentive compatible and (interim) individually rational for the firms. In that region find such a mechanism.

   Notice that this is *not* a symmetric problem, so the mechanism(s) need not be symmetric either.

Solution

By the Revelation Principle it is enough to consider direct mechanisms in which firms report their types (values). Given reports r, s of firms 1, 2 (respectively) the mechanism decides $\pi(r, s)$ (the probability of producing the project) and $x_1(r, s), x_2(r, s)$ (the payments of the two firms). Social efficiency means that $\pi(r, s) = 1$ if $r + s > c$ and $\pi(r, s) = 0$ if $r + s < c$ (ignoring ties). Individual rationality and no profit/no subsidy imply $x_1(0, s) = 0$ and $x_2(r, 0) = 0$ for all r, s.

(a) If $2 < c < 3$, it is socially efficient to build the project if and only if *both* firms have high valuation so set $\pi(1, 2) = 1$, $x_1(1, 2) = c/3$, $x_2(1, 2) = 2c/3$, $\pi(r, s) = x_1(r, s) = x_2(r, s) = 0$ for all $(r, s) \neq (1, 2)$. This is incentive compatible (because if either firm lies when it is a high valuation the project is not built but each firm wants the project built given the amount it is paying) and individually rational (because each firm either pays 0 or gets the project and pays less than its value).

(b) If $1 < c < 2$, it is socially efficient to build the project if and only if *firm 2* has high valuation (the valuation of firm 1 is irrelevant) set $\pi(1, 2) = \pi(0, 2) = 1$, $x_1 \equiv 0$, $x_2(1, 2) = x_2(0, 2) = c$, $\pi(r, s) = x_2(r, s) = 0$ for all $s \neq 2$.

(c) If $0 < c < 1$ it is socially efficient to build the project if and only if *either firm* has high valuation so we must have $\pi(0, 0) = 0$, $\pi(r, s) = 1$ otherwise. Now we have to worry about costs when $(r, s) \neq (0, 0)$.

IC for firm 1 when it has valuation 1: if it tells the truth the project is always built and it pays $x_1(1, 2)$ with probability p (firm 2 has high valuation) and $x_1(1, 0) = c$ with probability $1 - p$ (firm 2 has low valuation); if firm 1 lies the project is built when firm 2 has high valuation and not otherwise and firm 1 pays 0 always. So the IC constraint is

$$1 - px_1(1, 2) + (1 - p)c \geq p \cdot 1 \tag{1}$$

Similarly the IC constraint for firm 2 is

$$2 - px_2(1, 2) + (1 - p)c \geq p \cdot 2 \tag{2}$$

Add these and remember that $x_1(1, 2) + x_2(1, 2) = c$ to get

$$3 - pc + 2(1 - p)c \geq 3p$$

simplifying gives

$$p \leq (5/3)(1-c)$$

So outside of this region no such mechanism exists. Notice that if the constraints on c, p go in opposite directions: if p is small then it is dangerous to lie so you are willing to pay a lot when you tell the truth; if c is small then you don't pay much when you tell the truth so there is little incentive to lie.

Inside of this region we still have to find a mechanism which means find payments; any $x_1(1,2), x_2(1,2)$ that satisfy (1), (2) and

$$\begin{aligned} x_1(1,2) &\leq 1 \\ x_2(1,2) &\leq 1 \\ x_1(1,2) + x_2(1,2) &= c \end{aligned}$$

will work. [Notice that the individual rationality constraints are satisfied since firms pay 0 when they report 0 and pay no more than their true value in any case.]