Comprehensive Examination
Quantitative Methods
Spring, 2016

This exam consists of three parts. You are required to answer all the questions in all the parts.

Part I - 203A

Some potentially useful facts/hints can be found at the end.

Question 1 (5 pts.) *No derivation is required for this question; your derivation will not be read anyway.* Let $Z_1, Z_2, \ldots$ denote an i.i.d. sequence of standard normal random variables. Let

$$X_i = \begin{bmatrix} Z_i \\ Z_i^2 - 1 \end{bmatrix}.$$

What is the asymptotic distribution of $n^{-1/2} \sum_{i=1}^{n} X_i$ as $n \to \infty$? Your answer should be numerical; an abstract formula will not be accepted as an answer. (Also, you are required to be very specific about the dimension of the a zero vector or zero matrix. If you intend to write a 1×3 zero vector, you should write it $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ or $\underset{1\times 3}{0}$. If you simply write "0", it shall be understood to be a scalar.)

Question 2 *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that the joint PDF of random variables X and Y is given by

$$f_{Y,X}(y,x) = \frac{1}{2\sqrt{2\pi}} \frac{1(-|x| < y < |x|)}{|x|} \exp\left(-\frac{x^2}{2}\right),$$

where $1(-|x| < y < |x|)$ is the usual indicator function which is equal to 1 if $-|x| < y < |x|$ and 0 otherwise. (You may assume that $f_{Y,X}(y,x)$ takes an arbitrary value when $x = 0$, but it does not change anything in the sub-questions below.)

(a) (2 pts.) Calculate $E[Y^2 | x]$ at $x = \sqrt{3}$. Your answer should be a number.

(b) (1 pts.) Multiple Choice Question: Are X and Y independent of each other?
(A) They are independent.
(B) They are not independent.
(C) It is impossible to deduce given the set of available information.

(c) (2 pts.) Calculate $E[Y^2]$. Your answer should be a number.

Question 3 (5 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that X has a CDF $F(x)$ which is differentiable everywhere with strictly positive derivative, i.e., $F'(X) > 0$ for all x. Calculate $E[F(X)]$. Your answer should be a number.

Question 4 (5 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that

$$\begin{bmatrix} U \\ V \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

Let $M(t) = E\left[e^{tX}\right]$ denote the MGF of $X = UV$. Let $g(t) = dM(t)/dt$ and $h(t) = d^2 M(t)/dt^2$. Calculate $h(0) - (g(0))^2$. Your answer should be a number.

Question 5 (5 pts.) *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that $Z_1, Z_2, \ldots$ are i.i.d. $N(0,1)$. What is the probability limit of

$$\frac{1}{n}\sum_{i=1}^{n} Z_i \frac{1\left(0 < Z_i < 1\right)}{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{Z_i^2}{2}\right)}?$$

Here, the $1\left(0 < Z_i < 1\right)$ is the usual indicator function which is equal to 1 if $0 < Z_i < 1$, and 0 otherwise. Your answer should be a concrete number, not an abstract formula.

Question 6 (5 pts.) *In this question, your derivation/argument will be read and evaluated. That being said, if your ultimate answer is incorrect, your score for this question will be lower than or equal to 2 pts.* Consider the model

$$Y = \left\{ \begin{array}{ll} 1 & \text{if } \alpha + X_1 + X_2\beta_2 - \varepsilon \geq 0 \\ 0 & \text{otherwise} \end{array} \right\}$$

where $Y \in \{0,1\}$ and $X = (X_1, X_2, X_3)' \in R^3$ are observable, the distribution of ε conditional on $X = (x_1, x_2, x_3)$ is $N\left(0, \sigma\left(x_3\right)^2\right)$, $\sigma\left(\cdot\right)$ is an unknown positive-valued function. It is known that X has a discrete distribution with support

$$\left\{(x_1, x_2, x_3) : x_1, x_2, x_3 = 0, 1\right\},$$

i.e., its support is

$$\left\{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\right\}$$

What is the conditional probability of Y given $X = x$? Is $\sigma\left(\cdot\right)$ identified, i.e., are $\sigma\left(0\right), \sigma\left(1\right)$ identified? Is α identified? Is β_2 identified? Provide proofs for your answers. Your proof should state how identified parameters can be explicitly expressed as functions of $P\left(Y = 1 \middle| (x_1, x_2, x_3)\right)$, and $P\left(X = (x_1, x_2, x_3)\right)$ for $x_1, x_2, x_3 = 0, 1$.

Potentially Useful Facts/Hints:

1. If $Z \sim N(0,1)$, then $E\left[e^{tZ}\right] = \exp\left(t^2/2\right)$.
2. If Z has a $\Gamma(\alpha, \beta)$ distribution, i.e., if its PDF is given by

$$f(z) = \left\{ \begin{array}{ll} \dfrac{1}{\Gamma(\alpha)\,\beta^{\alpha}} z^{\alpha-1}e^{-z/\beta} & 0 < z < \infty \\ 0 & \text{elsewhere} \end{array} \right.$$

then $E\left[e^{tZ}\right] = \left(\dfrac{1}{1-\beta t}\right)^{\alpha}$ for $t < \frac{1}{\beta}$.

3. If $Z \sim \chi^2_r$, then $E\left[e^{tZ}\right] = (1-2t)^{-r/2}$.

Part II - 203B

Some potentially useful facts/hints can be found at the end.

Question 1 *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose we want to estimate a model

$$y_i = \beta_1 + \beta_2 \cdot x_{i2} + \beta_3 \cdot x_{i3} + \varepsilon_i, \quad i = 1, \ldots, n.$$

Suppose that the model satisfies the assumptions of classical linear regression model II. (See hints at the end in case you need to be reminded of the model II.) You estimated $(\beta_1, \beta_2, \beta_3)'$ by OLS. Your computer reported

$$\begin{bmatrix} \widehat{\beta}_1 \\ \widehat{\beta}_2 \\ \widehat{\beta}_3 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 0.6 \\ 0.32 \end{bmatrix}.$$

The number of observations (n) is equal to 27, and the sum of squared residuals ($e'e$) is equal to 0.2. Your computer also reported the estimated variance covariance matrix $\widehat{\mathbb{V}} = s^2 (X'X)^{-1}$ of $\left(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3\right)'$:

$$\begin{bmatrix} (0.4)^2 & \frac{1}{2}(0.4)(0.1) & \frac{1}{3}(0.4)(0.1) \\ \frac{1}{2}(0.4)(0.1) & (0.1)^2 & \frac{1}{2}(0.1)(0.1) \\ \frac{1}{3}(0.4)(0.1) & \frac{1}{2}(0.1)(0.1) & (0.1)^2 \end{bmatrix},$$

where s^2 is the unbiased estimate of $E[\varepsilon_i^2]$ based on $e'e$, as discussed in class.

(a) Test the hypothesis $H_0 : \beta_2 = 0.1$ against $H_1 : \beta_2 \neq 0.1$ under 5% significance level.

   i. (1 pt.) State the numerical value of your t-statistic.

   ii. (1 pt.) State the degrees of freedom of the t-distribution (the distribution of the t-statistic under the null). Your answer should be a concrete number.

   iii. (1 pt.) Multiple Choice Question: What is the result of your test? (Assume that the t-distribution is so close to the standard normal distribution that the difference can be ignored when characterizing the critical value.)
   (A) We reject the null.
   (B) We accept the null.
   (C) It is impossible to deduce given the set of available information.

(b) (2 pts.) Provide the 95% confidence interval for $\beta_2 + \beta_3$ using the formula you learned in class. (Again, ignore the difference of the t-distribution and the standard normal distribution.) Your answer should be numerical; an abstract formula will not be accepted as an answer.

(c) (3 pts.) Provide the asymptotic 95% confidence interval for β_2^2 (i.e., the square of β_2) using the delta-method based asymptotic formula that you learned in class. (Technically you should now assume that the model under consideration is the classical linear regression model III.) Your answer should be numerical; an abstract formula will not be accepted as an answer.

Question 2 *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that (y_i^*, x_i^*, y_i, x_i) $i = 1, 2, \ldots, n$ is an i.i.d. sequence with $y_i^* = x_i^*\beta + \varepsilon_i$ for $i = 1, \ldots, n$. Suppose that $y_i = y_i^* + \eta_i$ and $x_i = x_i^* + \nu_i$. Assume that $E[x_i^*\nu_i] = E[x_i^*\varepsilon_i] = E[x_i^*\eta_i] = 0$, $E[\nu_i\varepsilon_i] = E[\nu_i\eta_i] = 0$, and $E[\varepsilon_i\eta_i] = 0$. Also assume that $E\left[(x_i^*)^2\right] = 5$, $E[\varepsilon_i^2] = 4$, $E[v_i^2] = 3$, $E[\eta_i^2] = 2$. We will assume that $\beta \neq 0$ throughout.

(a) (2 pts.) Suppose that we only observe y_i and x_i^*, and define $\tilde{\beta}$ as the OLS estimate from regressing y_i on x_i^*, i.e., $\tilde{\beta} = \left(\sum_{i=1}^n x_i^* y_i\right) / \left(\sum_{i=1}^n (x_i^*)^2\right)$. What is $\left(\operatorname{plim}_{n\to\infty} \tilde{\beta}\right) \big/ \beta$? Your answer should be a number.

(b) (2 pts.) Suppose that we only observe y_i^* and x_i, and define $\hat{\beta}$ as the OLS estimate from regressing y_i^* on x_i, i.e., $\hat{\beta} = \left(\sum_{i=1}^n x_i y_i^*\right) / \left(\sum_{i=1}^n x_i^2\right)$. What is $\left(\operatorname{plim}_{n\to\infty} \hat{\beta}\right) \big/ \beta$? Your answer should be a number.

(c) (2 pts.) Multiple Choice Question: Suppose that we only observe y_i and x_i, and define $\bar{\beta}$ as the OLS estimate from regressing y_i on x_i, i.e., $\bar{\beta} = \left(\sum_{i=1}^n x_i y_i\right) / \left(\sum_{i=1}^n x_i^2\right)$. Which one of the following is correct? (A) $\operatorname{plim}_{n\to\infty} \bar{\beta} = \operatorname{plim}_{n\to\infty} \tilde{\beta}$; (B) $\operatorname{plim}_{n\to\infty} \bar{\beta} = \operatorname{plim}_{n\to\infty} \hat{\beta}$; (C) neither.

Question 3 *No derivation is required for the questions below; your derivation will not be read anyway.* Suppose that

$$y_i^* = x_i\beta + w_i\gamma + \varepsilon_i$$

such that (i) $E[x_i\varepsilon_i] = E[w_i\varepsilon_i] = 0$; (ii) (y_i, x_i, w_i) $i = 1, 2, \ldots, n$ are i.i.d.; (iii) the 2×2 matrix $E\left[(x_i, w_i)'(x_i, w_i)\right]$ is finite and positive definite; (iv) $\beta = 3$; and (v) $\gamma = 1$.

(a) (3 pts.) We will further assume that (i) $E[x_i] = E[w_i] = 0$; and (ii) x_i and w_i are independent of each other. If you regress y_i^* on x_i,what is the probability limit of the OLS coefficient as $n \to \infty$, i.e., what is the probability limit of $\left(\sum_{i=1}^n x_i y_i^*\right) / \left(\sum_{i=1}^n x_i^2\right)$. Your answer should be a number.

(b) (3 pts.) We will further assume that x_i, w_i, ε_i are all standard normal random variables independent of each other. Suppose that you observe $y_i = 1(y_i^* > 0)$ and x_i for each i, i.e., you do not observe w_i. (As usual, $1(y_i^* > 0)$ is an indicator function which is equal to 1 if $y_i^* > 0$ and 0 otherwise.) You decided to fit a probit model $\Pr[y_i = 1 | x_i] = \Phi(bx_i)$. What is the probability limit of the Probit MLE of b as $n \to \infty$? Your answer should be a number.

Question 4 (5 pts.) *No derivation is required for this question; your derivation will not be read anyway.* Suppose that Z_1 and Z_2 are i.i.d. and their common distribution is $N(0,1)$. What is $E[Z_1 | Z_1 > Z_2]$? Your answer should be a number, not an abstract formula.

Question 5 Suppose that our model is given by

$$\begin{aligned} y_i &= x_i\beta + \varepsilon_i \\ x_i &= z_i'\pi + v_i \end{aligned}$$

with the restriction

$$E[z_i\varepsilon_i] = E[z_i v_i] = 0.$$

We assume that $\dim(x_i) = \dim(\beta) = 1$, $\dim(z_i) = 3$, and $E\left[(z_i\varepsilon_i)(z_i\varepsilon_i)'\right] = \sigma_\varepsilon^2 E[z_i z_i']$. Assume further that $(x_i, z_i', \varepsilon_i)'$ $i = 1, 2, \ldots$ is an i.i.d. sequence, and that

$$\pi = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad E[z_i z_i'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \sigma_\varepsilon^2 = 2, \quad \beta \neq 0.$$

(a) (1 pt.) *No derivation is required for this question; your derivation will not be read anyway.* Let $\widehat{\beta}$denote 2SLS, i.e.,

$$\widehat{\beta} = \frac{\left(\sum_{i=1}^n x_i z_i'\right)\left(\sum_{i=1}^n z_i z_i'\right)^{-1}\left(\sum_{i=1}^n z_i y_i\right)}{\left(\sum_{i=1}^n x_i z_i'\right)\left(\sum_{i=1}^n z_i z_i'\right)^{-1}\left(\sum_{i=1}^n z_i x_i\right)}.$$

What is the asymptotic distribution of $\sqrt{n}\left(\widehat{\beta} - \beta\right)$ as $n \to \infty$? Your answer should be numerical; an abstract formula will not be accepted as an answer.

(b) (4 pts.) *In this question, your derivation/argument will be read and evaluated. That being said, if your ultimate answer is incorrect, your score for this question will be lower than or equal to 2 pts.* Let

$$\begin{aligned} \widetilde{\beta} &= \frac{\left(\sum_{i=1}^n y_i z_i'\right)\left(\sum_{i=1}^n z_i z_i'\right)^{-1}\left(\sum_{i=1}^n z_i y_i\right)}{\left(\sum_{i=1}^n y_i z_i'\right)\left(\sum_{i=1}^n z_i z_i'\right)^{-1}\left(\sum_{i=1}^n z_i x_i\right)} \\ &= \beta + \frac{\left(\sum_{i=1}^n y_i z_i'\right)\left(\sum_{i=1}^n z_i z_i'\right)^{-1}\left(\sum_{i=1}^n z_i \varepsilon_i\right)}{\left(\sum_{i=1}^n y_i z_i'\right)\left(\sum_{i=1}^n z_i z_i'\right)^{-1}\left(\sum_{i=1}^n z_i x_i\right)}. \end{aligned}$$

Derive the asymptotic distribution of $\sqrt{n}\left(\widetilde{\beta} - \beta\right)$, and compare it with that of $\sqrt{n}\left(\widehat{\beta} - \beta\right)$.

Potentially Useful Facts/Hints:

1. Classical linear regression **model II** is such that (i) $y = X\beta + \varepsilon$, where

$$\underset{n\times 1}{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad \underset{n\times k}{X} = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$$

(ii) X is a nonstochastic matrix; (iii) X has a full column rank; (iv) $\varepsilon \sim N\left(\underset{n\times 1}{0}, \sigma^2 I_n\right)$.

2. Classical linear regression **model III** is such that (i) $y_i = x_i'\beta + \varepsilon_i$; (ii) (x_i', ε_i) $i = 1, 2, \ldots$ is i.i.d.; (iii) x_i is independent of ε_i; (iv) $E[\varepsilon_i] = 0$, $\text{Var}(\varepsilon_i) = \sigma^2$; (v) $E[x_i x_i']$ is positive definite and finite.

3. In the sample selection model

$$\begin{aligned} y_i^* &= x_i'\beta + u_i \\ D_i &= 1\left(z_i'\gamma + v_i > 0\right) \\ y_i &= D_i \cdot y_i^* \end{aligned}$$

with

$$\left(\begin{array}{c} u_i \\ v_i \end{array}\right)\Bigg| (x_i, z_i) \sim N\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left[\begin{array}{cc} \sigma_u^2 & \rho\sigma_u\sigma_v \\ \rho\sigma_u\sigma_v & \sigma_v^2 \end{array}\right]\right),$$

it was shown in class that

$$E\left[y_i | x_i, z_i, D_i = 1\right] = x_i'\beta + \rho\sigma_u \cdot \frac{\phi\left(z_i'\frac{\gamma}{\sigma_v}\right)}{\Phi\left(z_i'\frac{\gamma}{\sigma_v}\right)}.$$

4. In a linear model

$$y_i = x_i' \underset{m\times 1}{\beta} + \varepsilon_i$$

with the restriction that

$$E\left[\underset{k\times 1}{z_i}\ \varepsilon_i\right] = 0$$

where $k \geq m$, the **2SLS**

$$\widehat{\beta}_{2SLS} = \left[\left(\sum_i x_i z_i'\right)\left(\sum_i z_i z_i'\right)^{-1}\left(\sum_i z_i x_i'\right)\right]^{-1}\left(\sum_i x_i z_i'\right)\left(\sum_i z_i z_i'\right)^{-1}\left(\sum_i z_i \cdot y_i\right)$$

is such that under standard assumptions, $\sqrt{n}\left(\widehat{\beta}_{2SLS} - \beta\right)$ converges in distribution to

$$N\left(0, \sigma_\varepsilon^2\left[E\left[x_i z_i'\right]\left(E\left[z_i z_i'\right]\right)^{-1}\left(E\left[z_i x_i'\right]\right)\right]^{-1}\right)$$

if $E\left[(z_i\varepsilon_i)(z_i\varepsilon_i)'\right] = \sigma_\varepsilon^2 E\left[z_i z_i'\right]$, where $\sigma_\varepsilon^2 = E\left[\varepsilon_i^2\right]$.

5. In **Question 5(b)**, we can write $\widetilde{\beta} = 1/\widetilde{\delta}$, $\beta = 1/\delta$, where

$$\begin{aligned} x_i &= \delta y_i + u_i, \\ u_i &= -\frac{\varepsilon_i}{\beta}, \\ E\left[z_i u_i\right] &= 0, \end{aligned}$$

and

$$\widetilde{\delta} = \frac{\left(\sum_{i=1}^n y_i z_i'\right)\left(\sum_{i=1}^n z_i z_i'\right)^{-1}\left(\sum_{i=1}^n z_i x_i\right)}{\left(\sum_{i=1}^n y_i z_i'\right)\left(\sum_{i=1}^n z_i z_i'\right)^{-1}\left(\sum_{i=1}^n z_i y_i\right)}.$$

Part III - 203C

Some useful lemmas and theorems can be found at the end.

Question 1 (6 pts.) Let $I\{\cdot\}$ denote the indicator function. Suppose that we have one observation from a probability density function $f_\theta(x)$. Find a uniformly most powerful test of size $\alpha = 0.01$ (with explicit critical region) for

$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta = \theta_1$$

when

(a) (3 pts.) $f_\theta(x) = 2\theta^{-2}(\theta - x)I\{0 \leq x \leq \theta\}$, $\theta_0 = 1$ and $\theta_1 = 2$;

(b) (3 pts.) $f_{\theta_0}(x) = 4xI\{0 \leq x \leq \frac{1}{2}\} + 4(1-x)I\{\frac{1}{2} \leq x \leq 1\}$ and $f_{\theta_1}(x) = I\{0 \leq x \leq 1\}$.

Question 2 (24 pts.) Suppose that $\{X_t\}_t$ and $\{Y_t\}_t$ are covariance stationary processes satisfying

$$\begin{aligned} X_t - \theta_0 X_{t-1} &= u_t, \\ Y_t - \theta_0 Y_{t-1} &= X_t + v_t, \end{aligned}$$

where $|\theta_0| < 1$, $\{(u_t, v_t)'\}_t$ is an i.i.d. process, i.e., $\{(u_t, v_t)'\}_t \sim i.i.d.(0, \Sigma)$, where

$$\Sigma = \begin{pmatrix} \sigma_u^2 & \sigma_{u,v} \\ \sigma_{u,v} & \sigma_v^2 \end{pmatrix}.$$

(a) (4.5 pts.) Show that $\sum_{h=-\infty}^{\infty} |\Gamma_Y(h)| < \infty$, where $\Gamma_Y(h)$ denotes the autocovariance function of $\{Y_t\}_t$.

(b) (4.5 pts.) Find the spectral density of Y_t. Depict the spectral density of Y_t.

(c) (3 pts.) Suppose that $E[u_1^4] < \infty$ and we have data $\{X_t\}_{t=1}^T$. Construct a consistent estimator of the long-run variance of $\{X_t\}_t$. Show the consistency of your estimator.

(d) (1.5 pts.) Show that θ_0 is the unique solution of the linear equations $G_1(\theta) = (0, 0)'$, where

$$G_1(\theta) = E\left[\begin{array}{c} (X_t - \theta X_{t-1})X_{t-1} \\ (Y_t - \theta Y_{t-1} - X_t)Y_{t-1} \end{array}\right].$$

(e) (3 pts.) Suppose that $\sigma_{u,v} = 0$, $E[u_1^4 + v_1^4] < \infty$ and we have data $\{(X_t, Y_t)\}_{t=1}^T$. Construct a consistent estimator of the efficient weight matrix of the GMM estimator based on the moment conditions $G_1(\theta_0) = (0, 0)'$. Show the consistency of your estimator.

(f) (4.5 pts.) Suppose that $\sigma_{u,v} = 0$. Let $V_{1,\theta}^*$ denote the asymptotic variance of the efficiently weighted GMM estimator based on the moment conditions $G_1(\theta_0) = (0, 0)'$. Find the explicit form of $V_{1,\theta}^*$. Is $V_{1,\theta}^*$ strictly smaller than the asymptotic variance of the LS estimator $\widehat{\theta}_{LS} = \sum_{t=2}^T X_{t-1}X_t \Big/ \sum_{t=2}^T X_{t-1}^2$? Justify your answer.

(g) (3 pts.) Suppose that $\sigma_{u,v} = 0$. Let $V^*_{2,\theta}$ denote the asymptotic variance of the efficiently weighted GMM estimator based on the moment conditions $G_2(\theta_0) = (0,0,0,0)'$, where

$$G_2(\theta) = E\begin{bmatrix} (X_t - \theta X_{t-1})X_{t-1} \\ (X_t - \theta X_{t-1})Y_{t-1} \\ (Y_t - \theta Y_{t-1} - X_t)Y_{t-1} \\ (Y_t - \theta Y_{t-1} - X_t)X_{t-1} \end{bmatrix}.$$

Find the explicit form of $V^*_{2,\theta}$. Is $V^*_{2,\theta}$ strictly smaller than $V^*_{1,\theta}$? Justify your answer.

Potentially Useful Theorems and Lemmas:

Theorem 1 (Martingale Convergence Theorem) *Let $\{(X_t, \mathcal{F}_t)\}_{t\in\mathbb{Z}_+}$ be a martingale in L^2. If $\sup_t E\left[|X_t|^2\right] < \infty$, then $X_n \to X_\infty$ almost surely, where X_∞ is some element in L^2.*

Theorem 2 (Martingale CLT) *Let $\{X_{t,n}, \mathcal{F}_{t,n}\}$ be a martingale difference array such that $E[|X_{t,n}|^{2+\delta}] < \Delta < \infty$ for some $\delta > 0$ and for all t and n. If $\overline{\sigma}^2_n > \delta_1 > 0$ for all n sufficiently large and $\frac{1}{n}\sum_{t=1}^n X^2_{t,n} - \overline{\sigma}^2_n \to_p 0$, then $n^{\frac{1}{2}}\overline{X}_n/\overline{\sigma}_n \to_d N(0,1)$.*

Theorem 3 (LLN of Linear Processes) *Suppose that Z_t is i.i.d. with mean zero and $E[|Z_0|] < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k\,|\varphi_k| < \infty$. Then $n^{-1}\sum_{t=1}^n X_t \to_{a.s.} 0$.*

Theorem 4 (CLT of Linear Processes) *Suppose that Z_t is i.i.d. with mean zero and $E[Z_0^2] = \sigma_Z^2 < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k^2\varphi_k^2 < \infty$. Then $n^{-\frac{1}{2}}\sum_{t=1}^n X_t \to_d N\left[0, \varphi(1)^2\sigma_Z^2\right]$.*

Theorem 5 (LLN of Sample Variance) *Suppose that Z_t is i.i.d. with mean zero and $E[Z_0^2] = \sigma_Z^2 < \infty$. Let $X_t = \sum_{k=0}^\infty \varphi_k Z_{t-k}$, where φ_k is a sequence of real numbers with $\sum_{k=0}^\infty k\varphi_k^2 < \infty$. Then*

$$\frac{1}{n}\sum_{t=1}^n X_t X_{t-h} \to_p \Gamma_X(h) = E\left[X_t X_{t-h}\right]. \tag{1}$$

Theorem 6 (Donsker) *Let $\{u_t\}$ be a sequence of random variables generated by $u_t = \sum_{k=0}^\infty \varphi_k \varepsilon_{t-k} = \varphi(L)\varepsilon_t$, where $\{\varepsilon_t\} \sim iid\ (0, \sigma^2_\varepsilon)$ with finite fourth moment and $\{\varphi_k\}$ is a sequence of constants with $\sum_{k=0}^\infty k\,|\varphi_k| < \infty$. Then $B_{u,n}(\cdot) = n^{-\frac{1}{2}}\sum_{t=1}^{[n\cdot]} u_t \to_d \lambda B(\cdot)$, where $\lambda = \sigma_\varepsilon\varphi(1)$.*