Question 1

Consider an economy populated by a continuum of measure one of consumers whose preferences are defined by the utility function:

$$\sum_{t=0}^{\infty} \beta^t \log(C_t),$$

where C_t is consumption and the parameter β satisfies $0 < \beta < 1$. Each consumer provides one unit of labor inelastically.

The aggregate production technology is given by:

$$Y_t = AK_t^{\alpha} N_t^{1-\alpha} G_t^{1-\alpha},$$

where K_t is business capital, N_t is aggregate labor supply, G_t is public capital (i.e., public infrastructure such as roads), the parameter α satisfies $0 < \alpha < 1$, and $A > 0$ is constant. Both types of capital depreciate completely every period. Output can be transformed one-for-one into consumption goods, new business capital, or new public capital.

Part A: Provide a Bellman equation for the social planning problem. Can the problem be expressed using a single state variable? Do the Blackwell conditions apply to this Bellman equation? Which ratio of production capital and infrastructure capital K_{t+1}/G_{t+1} will be chosen by the planner? What is the growth rate of output and consumption chosen by the planner? (*For the last parts, you can make direct use of first-order and envelope conditions; you are not required to prove existence and uniqueness of a fixed point for the Bellman equation first*)

Part B: We will now compare the planning outcome to the outcome in a competitive economy in which public capital is financed through taxes. Firms are renting physical capital and labor at rates r_t and w_t from households, respectively, while public capital is provided at no cost for firms. The government levies a constant proportional income tax τ on households to finance infrastructure investment. The household's budget constraint is:

$$C_t + K_{t+1} = (1 - \tau)(r_t K_t + w_t),$$

and infrastructure investment is given by:

$$G_{t+1} = \tau(r_t K_t + w_t).$$

Define a sequence-of-markets equilibrium. What is the balanced growth rate of this economy? Can the government choose a τ that reproduces the social planning outcome? If not, is there an alternative arrangement for financing infrastructure that could achieve the optimal outcome?

Question 2

Find the equilibrium solution to the log-linearized first order conditions of the following stochastic growth model. Assume that consumers have preferences of the form

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

with

$$u(c) = \frac{1}{1-\sigma} c^{1-\sigma}$$

The resource constraint is given by

$$c_t + k_t - (1-\delta)k_{t-1} = A_t k_{t-1}^{\alpha}$$

with A_t representing the current value of Total Factor Productivity at t. Assume that the logarithm of A_t evolved according to

$$\log(A_{t+1}) = (t+1)\mu + \rho(\log(A_t) - t\mu) + \epsilon_{t+1}$$

with ϵ_{t+1} drawn independently each period from a Normal Distribution with mean zero and variance σ^2.

Assume that $\beta, \rho, \alpha \in (0,1)$ and $\sigma > 0$.

Part A: Over the long run, what is the average growth rate of output in this economy in terms of the parameters above? Choose a transformation of the variables in this economy that will render the variables stationary and write down the equations of the model that need to be log-linearized in terms of these variables.

Part B: Write down the log-linearized version of these equations.

Part B: Find a solution to the model of the form

$$\hat{k}_t = P\hat{k}_{t-1} + Q\hat{A}_t$$

$$\hat{c}_t = R_c\hat{k}_{t-1} + S_c\hat{A}_t$$

In particular, solve for P, Q, R_c, S_c, explicitly in terms of the parameters above and the steady-state fractions of output consumed $\frac{\bar{c}}{\bar{Y}}$, the capital-output ratio $\frac{\bar{k}}{\bar{Y}}$.

Question 3

Consider an overlapping generations model in which time is denoted $t = 1, 2, 3, \ldots$. Every period t, a new generation is born and lives in periods t and $t+1$. At date $t = 1$ there is an initial generation of old agents who live only at that date. The size of the population of each generation of agents is constant and fixed at 1.

At each date t, the consumption of agents born at t (the young) is denoted c_t^y while the consumption of agents born in the previous period (the old or the initial generation at $t = 1$) is denoted c_t^o. Agents born at t have preferences over consumption at t and $t+1$ given by

$$\log(c_t^y) + \log(c_{t+1}^o)$$

while the initial generation has preferences given by

$$\log(c_1^o)$$

Each agent born at t is endowed with one unit of output when young and ϵ units of output when old. Thus, the aggregate endowment is $1 + \epsilon$ every period. Also assume that the initial old are endowed with M units of paper money.

Part A: Define a competitive equilibrium in this environment, taking care to include the possibility that the initial stock of money M has positive value and the possibility that it does not.

Part B: Assume that $\epsilon > 1$. Characterize all of the competitive equilibria. Are these equilibria all Pareto Optimal?

Part C: Assume that $\epsilon < 1$. Describe all of the deterministic competitive equilibria. Are all of these equilibria Pareto Optimal? Can these equilibria be Pareto ranked?

Question 4

Consider a model in which time is discrete and denoted by $t = 0, 1, 2, 3, \ldots$. There is a representative household with preferences over sequences of consumption $\{c_t\}$ given by

$$\sum_{t=0}^{\infty} \beta^t \log(c_t)$$

This household has one unit of labor that it supplies inelastically to the market every period.

Output is produced in a continuum of heterogeneous competitive firms. Each firm is indexed by its current level of productivity z. A firm with productivity z can produce output with labor according to

$$y = z^{(1-\nu)} l^{\nu}$$

At date $t = 0$, the measure of firms is given by N_0 and the density of productivities across firms is given by $f_0(z)$.

Existing firms in this economy solve a static profit maximization problem, choosing employment $l_t(z)$ to solve

$$\max_{l} z^{(1-\nu)} l^{\nu} - w_t l$$

Let the maximized profits at t (given wage w_t) be denoted $\pi_t(z)$.

For part A of this problem, assume that the measure of firms remains constant over time, with $N_t = N_0$ for all t.

Let $l_t(z)$ denote the hiring decision of a firm with productivity z at t. Aggregate output is given by

$$y_t = N_t \int_z z^{(1-\nu)} l_t(z)^{\nu} f_t(z) dz$$

Goods market clearing requires $c_t = y_t$.

Labor market clearing requires that

$$N_t \int_z l_t(z) f_t(z) dz + M_t n_e = 1$$

An equilibrium in this economy is a collection of sequences $\{c_t, y_t, l_t(z), N_{t+1}\}$, prices $\{p_t, w_t\}_{t=0}^{\infty}$, and firm profits $\{\pi_t(z)\}$ such that the representative household is choosing $\{c_t\}$ to maximize its utility subject to a date zero budget constraint

$$\sum_{t=0}^{\infty} p_t \left[w_t + N_t \int_z \pi_t(z) f_t(z) dz - c_t \right] \geq 0,$$

and firms are choosing labor $l_t(z)$ to maximize profits.

Part A: Show how to reduce the problem of finding equilibrium to one of finding aggregate sequences $\{c_t, y_t, N_{t+1}\}$, and prices $\{p_t, w_t\}_{t=0}^{\infty}$. To do so, you will need to aggregate across firms to construct a measure of aggregate productivity Z_t and describe the evolution of this aggregate productivity variable over time. In particular, compute Z_t explicitly.

Now consider a version of this economy in which at the end of each period t, each firm either dies with probability δ or continues next period with the same productivity z with probability $(1-\delta)$.

New firms can be created with an expenditure of n_e units of labor per new firm. Specifically, to create M_t new firms starting at $t+1$, $M_t n_e$ units of labor are used in period t. These new firms have productivity drawn from density $g(z)$.

There is free entry into the business of starting a new firm. We let p_t denote the price of a unit of consumption at t relative to consumption at $t=0$. There are zero profits to entry if and only if

$$n_e w_t = \sum_{k=1}^{\infty} \frac{p_{t+k}}{p_t}(1-\delta)^{k-1} \int_z \pi_{t+k}(z) g(z) dz$$

We require that there be zero profits to entry at t if $M_t > 0$ and non-positive profits to entry at t if $M_t = 0$.

The evolution of the number of firms is given by

$$N_{t+1} = (1-\delta)N_t + M_t$$

and the evolution of the distribution of productivities across firms is given implicitly by

$$f_{t+1}(z)N_{t+1} = (1-\delta)f_t(z)N_t + g(z)M_t$$

An equilibrium in this economy is a collection of sequences $\{c_t, y_t, l_t(z), N_{t+1}, M_t\}$, prices $\{p_t, w_t\}_{t=0}^{\infty}$, and firm profits $\{\pi_t(z)\}$ such that the representative household is choosing $\{c_t\}$ to maximize its utility subject to a date zero budget constraint

$$\sum_{t=0}^{\infty} p_t \left[w_t + N_t \int_z \pi_t(z) f_t(z) dz - c_t \right] \geq 0,$$

firms are choosing labor $l_t(z)$ to maximize profits, there are zero profits to entry at t if $M_t > 0$ and non-positive profits if $M_t = 0$, and goods and labor markets clear.

Part B: Compute the steady-state value of aggregate productivity Z_t.

Question 5

A Ramsey Planning Problem

Consider a closed economy with a measure one continuum of households. Time is discrete and runs from 0 to ∞. The households evaluate a sequence of consumption and labor supply choices allocations and labor supply choices $\{c_t, n_t\}_{t=0}^{\infty}$ according to the utility functional

$$U = \sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) - v\left(n_t\right)\right].$$

Output y is produced using labor n and capital k as inputs, according to a technology $y = f(k, n)$, and can be used as consumption, government expenditure g_t, as well as new capital. Capital depreciates at a rate δ, so that in each period,

$$f\left(k_t, n_t\right) = y_t \geq c_t + g_t + k_{t+1} - (1-\delta) k_t$$

Households can trade in one period riskless bonds. Assume that $g_t = g > 0$ is exogenous, and that there is no uncertainty in this economy. The government can freely choose labor and capital taxes in all periods, except date 0. The government's objective is given by

$$\hat{U} = \sum_{t=0}^{\infty} \hat{\beta}^t \left[u\left(c_t\right) - v\left(n_t\right)\right],$$

where $\hat{\beta}$ may or may not be equal to the households' discount rate β.

(i) Formulate the corresponding Ramsey Planning problem.

(ii) Assume that $\beta = \hat{\beta}$. Characterize the solution to this problem, and explain its main features. What implications can you draw for capital taxes? What implications can you draw for labor taxes?

(iii) Assume next that $\beta \neq \hat{\beta}$. Again characterize the solution to the extent possible. How does a difference in discount rate affect your results compared to (ii)? (Remark: if you feel comfortable doing so, you may combine your analysis in parts (ii) and (iii) by treating the case $\beta = \hat{\beta}$ as a special case of the more general formulation).

(iv) Can you offer an economic interpretation for the assumption that $\hat{\beta} > \beta$ (i.e. that the planner is more patient than households)? What about the opposite case, $\hat{\beta} < \beta$? How does this help you interpret the optimal tax results you found under (iii)?

Question 6

Redistribution with Limited Commitment

Consider the following version of a social planning model with limited commitment: There is a household, with preferences given by:

$$U_t = \sum_{s=0}^{\infty} \beta^s \log(c_{t+s})$$

In each period t, the householdreceives a stochastic income e_t , which is independent over time, and distributed as follows:

$$e_t = \begin{cases} y + \varepsilon \text{ w.p. } 1/2 \\ 1 + \varepsilon - y \text{ w.p. } 1/2 \end{cases},$$

where $y \in (1/2, 1)$, and $\varepsilon > 0$.

The household seeks insurance from a risk-neutral planner who has the ability to borrow or lend externally at a rate $R < \beta^{-1}$. At any point in time, however, the household can decide to default on his contractual obligations. After a default, the household is left in autarky, and in all subsequent periods the planner can confiscate ε from the household's income (i.e. the income after default is reduced by ε in all future periods).

(i) Characterize the solution to the planner problem assuming full commitment (i.e. assuming that the power to confiscate income after a default is unlimited).

(ii) Characterize the solution to the planning problem with limited commitment. When is the full commitment solution attainable?

(iii) Suppose now that a well-meaning government comes into power, and upon observing the very large degree in income fluctuations, implements policies that reduce the variation in income (as measured in our model by y). This reduction in income fluctuations applies to all agents equally, irrespective of their default status. Explain, using your limited commitment model, how the level of risk sharing is affected by the change in income fluctuations. In particular, show that a decrease in income variation may lead to an increase in consumption volatility.