Uncertainty, Information and Games

Option 1: Answer 4 questions from sections A and B.
Option 2: Answer 4 questions, at least one from each of the three sections.

Section A.

1. Screening

A type t worker has a productivity $V(t) = t$, where t is uniformly distributed on [0,1]. His cost of education level z is $C(t,z) = z/t^a$. A large number of firms which compete for his services. The worker's type is private information.

(a) Solve for the Pareto-preferred separating wage function $w^*(z)$ and hence show that the resulting cost of education is $C(t,z^*(t)) = t/(1+\alpha)$.
(b) What is the resulting cost of education if instead workers can choose an education level z at cost of $C(t,z) = g(z)/t^a$, where $g(0) = 0$ and $g(\cdot)$ is strictly increasing?
(c) By considering offers at $z = 0$ (or otherwise) find a necessary condition for the solution to part (a) to be a Nash equilibrium.
(a) How you would go about obtaining sufficient conditions for a Nash equilibrium.

2. Signaling

There are 4 types of consultant offering their services to a large number of firms. The probability that an individual is of type t is p_t, the value of the services offered by type t is V_t. The cost to type t of sending a signal z is $C_t(z)$. Any information an individual has about his type is private. All of the following data is common knowledge.

t	p_t	$C_t(z)$	$V_t(z)$
1A	1/6	z	0
1B	1/12	z	3
2A	¼	z/2	0
2B	½	z/2	3

Initially suppose that each consultant knows if he is a 1 or a 2 but not whether he is type A or type B.

(a) Show that the expected productivity of a type 1 individual is 1 and of a type 2 is 2 and hence show that there is a continuum of separating Nash equilibria in which wages are 1 and 2..
(b) Define the "Intuitive Criterion" and show that there is a unique separating Nash equilibrium which satisfies this refinement.
(c) Does this equilibrium also satisfy the Grossman-Perry refinement?

Henceforth assume that each individual has full information about his own type.

(d) Which of the Nash separating equilibria in (a) satisfy the Intuitive Criterion?
(e) Is there a Nash equilibrium which satisfies the Grossman-Perry refinement.

3. **Optimal Auctions**

Bidder *i*, *i*=1,2 has a private valuation which is a draw from a continuous distribution with c.d.f $F_i(v)$ and support $[\alpha_i, \beta_i]$, where $\beta_1 \geq \beta_2 > \alpha_1 \geq \alpha_2$. The auctioneer designs an auction in which, if both bidders use their equilibrium best responses, bidder *i*'s probability of winning is $\pi_i(v_1, v_2)$. Let $U_i^*(v_i)$ be the equilibrium expected payoff to bidder *i* if he is of type v_i.

(a) Show that the equilibrium expected payoff for bidder *i* is

$$\bar{U}_i^* = U_i^*(\alpha_i) + \iint_{v_1, v_2} \pi_i(v_1, v_2) \frac{1 - F_i(v_i)}{F_i'(v_i)} dF_1 dF_2 .$$

(b) Hence obtain an expression for the equilibrium expected revenue.

For the next two parts, assume that $F_i(v) = \frac{\alpha_i + v}{\alpha_i + \beta}$, where $\beta > \alpha_i > 0$. The seller is committed to using a second-price auction with reserve prices.

(c) What reserve price or prices will he set for the two buyers?
(d) Are there other selling mechanisms which yield a higher equilibrium expected revenue?
(e) How would your answers to parts (c) and (d) change if instead,

$$F_i(v_i) = \frac{v_i - \alpha}{\beta_i - \alpha}, \text{ where } \beta_i > \alpha > 0$$

Section B.

4. Team Incentives and Tournaments:

A risk-neutral principal hires two workers, $i = 1,2$, each exerting unobserved effort e_i with cost of $e_i^2 / 2$. Two output signals are verifiable: $x_i = e_i + \varepsilon_i + \varepsilon_0$, where for $i = 1,2$, $\varepsilon_i \propto N(0,1)$, $\varepsilon_0 \propto N(0, \sigma^2)$, and ε_o, ε_1 and ε_2 are independent. Worker i's utility function is given by $u_i = -e^{-r[s_i(x_i) - e_i^2/2]}$. We only consider linear wage contracts in this question. Both workers must get a certainty equivalent of zero, otherwise they will quit.

(a) Suppose worker i's wage contract can only depend on x_i. Derive the optimal linear contract for worker i. What is the total expected net profit for the principal (using symmetry)? [Hint: remember that $\varepsilon_0 + \varepsilon_1 \propto N(0, 1 + \sigma^2)$]
(b) Now suppose bother workers' wage contracts can only depend on $x_1 - x_2$. Derive the optimal linear contracts. What is the total expected net profit for the principal? [Hint: remember that $\varepsilon_1 - \varepsilon_2 \propto N(0, 2)$]
(c) Compare (a) and (b), how does this comparison vary with σ^2? What is the economic meaning of this result?
(d) Now suppose the only signal the principal can observe is whether $x_1 - x_2$ is positive. So she designs a tournament: the winner gets W and the loser gets w. Sketch how the

principal should set the two wages. Is this possible the two workers work harder under the optimal tournament than in (a) or (b)?

5. Optimal Mechanism with Adverse Selection and Moral Hazard:

A firm hires a risk-neutral researcher to develop a patent. The value of the patent is $x = ke + \varepsilon$, where ε is a mean zero noise, $k \geq 1$ is a constant and e is the researcher's unobservable effort. The effort cost to the researcher is $C = (2-t)e^2 / 2$, where t is the researcher's type. The firm does not observe the researcher's type but knows it is uniformly distributed on [0,1]. The researcher's reservation utility is zero.

(a) Let the firm's wage contract be $w(\hat{t}, x)$ and effort recommendation $e(\hat{t})$. Write down the optimal mechanism design problem.
(b) For any effort $e(t)$, to induce truth-telling, show that the expected utility of the researcher is given by $U(t) = U(0) + \int_0^t 0.5e^2(s)ds$
(c) Solve for the optimal effort that maximizes the firm's expected profit.
(d) Show that a wage contract linear in x implements the optimal effort found in (c).

6. Hold-up Problem with Multi-dimensional Investments

An upstream firm is run by a manager M1 and a down stream firm is run by a manager M2. The total value of their trade depends on the relation-specific investments made by the two managers. Specifically, suppose $x = e_1 + e_2$. In addition, each manager can also exert efforts in improving his outside option. Let manager 1's outside option be $y_1 = \alpha r_1$ where r_1 is his investment in outside option and $0 < \alpha < 1$ is a positive constant. Similarly, let manager 2's outside option be $y_2 = \beta r_2$ where r_2 is her investment in outside option and $0 < \beta < 1$ is a positive constant. Manager i's investment cost is $c_i = (e_i + r_i)^2 / 2$.

(a) Derive the first best investment decisions.
(b) Suppose investments are not contractible, nor is total profit. The two managers first make investments simultaneously, then bargain over the division of the total profit according to the Nash Bargaining Solution. Derive the equilibrium outcome.
(c) Now suppose there is a physical asset. Without owning the asset, a manager cannot obtain any outside option. Derive the total surplus under manager 1 ownership and manager 2 ownership. What implications can we get from comparing the total surpluses? What is the optimal ownership structure?
(d) Suppose the investment costs change to $c_i = e_i^2 / 2 + r_i^2 / 2 - ke_i r_i$. How would your answer to (c) change?

Section C

(7) Competition with moral hazard

Consider an economy with a continuum of agents (workers), where one good is produced by many competing production firms. The production technology is such that each worker employed by a firm can supply either high effort, e_h, or low effort, e_l, in production (with $e_l < e_h$). y^a and y^b (with $y^a > y^b$) are the possible levels of production for each worker; P is the probability that a worker who has exerted the high level of effort produces y_a, while p is the probability that a worker who has exerted the low level of effort produces y_a, (with $P > p$). The technology is linear with respect to the number of workers hired. The utility function of the agent is $u(c, e)$ where c denotes agent's consumption and e the effort exerted. $u(c, e)$ is strictly concave in c and n-times differentiable. A labor contract, (w_a, w_b), specifies the wage, w_o contingent on y_o $(o = a, b)$. Risk neutral firms compete by offering exclusive labor contracts. Workers cannot trade in the insurance market.

In answering to questions **(a)** and **(b)** assume that in equilibrium all agents make high effort.

(a) Characterize the equilibrium contract.

(b) Suppose that both P and p increase by ε_1, and ε_2 respectively, with ε_1 and ε_2 small (i.e. the productivity of the agent conditional on low and high effort increases). Show that there exist positive values of ε_1 and ε_2 such that the agents' equilibrium expected utility decreases. Explain the intuition.

(c) Show (graphically or analytically) that the contract characterized in answering to question **(a)** is not offered when agents can also trade fair contracts in the insurance markets and insurance trades are anonymous. Assume for simplicity that preferences are separable in consumption and effort. Discuss the economic intuition.

(d) Suppose that agents can conclude fair insurance contracts in the insurance market after having being employed by some firm. Characterize the equilibrium contract offered by the firms.

Suppose now that firms can only employ $1/2$ of the workers and that U is the workers' reservation utility. Assume also that, as in questions **(a)** and **(b)**, agents cannot trade in the insurance market.

(e) Provide conditions on preferences and technologies such that the workers who are employed get, in equilibrium, an expected utility higher than U.

(8) Signalling and adverse selection in the credit market

Consider an economy with a large number of entrepreneurs who can undertake a project that yields revenue Y with probability θ and 0 with probability $1-\theta$. The project has a fixed size and need an investment of 1 to be implemented. A debt contract (t, C) specifies a reimbursement t to the lender if the project is successful and an amount of collateral C to be paid if the project fails. C has to be greater or equal than zero. The project involves a fixed nonmonetary cost b for each entrepreneur. The entrepreneur's expected utility is 0 if he does not borrow and $u_e(t, C) = \theta(R-t) - (1-\theta)C - b$ if he borrows. The lender values C units of collateral αC, with $\alpha < 1$. He is risk neutral and maximizes expected profit. θ takes values $\bar{\theta}$ with probability p and $\hat{\theta}$ with probability $(1-p)$, (with $\bar{\theta} > \hat{\theta}$). The distribution of θ is common knowledge.

Suppose first there is a single lender who can offer debt contracts after having learned the true value of θ of each entrepreneur, while entrepreneurs do not know their θ when they trade with the lender.

(a) Characterize the couple of separating contracts satisfying the participation constraints preferred by the lender.

Suppose now that each entrepreneur knows his own θ before trading, while the lender only knows the distribution of θ.

(b) Characterize the couple of separating contracts preferred by the lender.

(c) Suppose that the lender wants to lend to all agents (i.e. whatever is the realization of θ). Does he offer separating or pooling contracts?

Finally, suppose that there are many competing lenders and only the entrepreneurs know their own θ. Each lender can offer at most one contract. In a first stage contracts are offered simultaneously, then entrepreneurs choose which contract to take. Contracts cannot be retired. Suppose that an equilibrium exists.

(d) Characterize the equilibrium contracts (just write the conditions that equilibrium contract has to satisfy, do not solve explicitly).