ANSWERS

**Answer for Q1**

(a) The budget constraint can be written as $p \cdot (a_i \mathbf{1} + x_i) \leq p \cdot (a_i \mathbf{1} + e_i)$. So, assuming an interior solution, the demand function is given by $x_{i,\ell}(p, e_i) = \frac{\pi_\ell p \cdot (a_i \mathbf{1} + e_i)}{p_\ell} - a_i$.

(b) So $\sum_{i=1}^{I} x_{i,\ell}(p, e_i) = \frac{\pi_\ell p \cdot (A \mathbf{1} + E)}{p_\ell} - A$, where $A = \sum_{i=1}^{I} a_i$ and $E = \sum_{i=1}^{I} e_i$. This can be regarded as the demand of the representative consumer with utility function $\sum_{\ell=1}^{L} \pi_\ell \log(A + x_\ell)$ and budget constraint $p \cdot x \leq p \cdot E$ (again with an interior solution).

(c) In equilibrium, $A + E_\ell = \frac{\pi_\ell p \cdot (a_i \mathbf{1} + e_i)}{p_\ell}$ for $\ell = 1, 2$. Taking the ratio, we obtain the equilibrium price ratio $\frac{p_2^*}{p_1^*} = \frac{\pi_2}{\pi_1} \frac{A+E_1}{A+E_2} = \frac{1}{2}$. Hence $x_{i,1}^* = \frac{3a_i + 85}{4} - a_i$ and $x_{i,2}^* = \frac{3a_i+85}{2} - a_i$.

(d) When $a_1 = a_2 = 0$, Arrow-Debreu equilibrium generates no trade. Hence no asset would be needed to generate it. When $a_1 = 4$, $a_2 = 6$, the consumers need to trade to achieve A-D equilibrium outcome. However, there is only one asset in this economy as $e_1 = e_2$ (hence a share of each plantation is exactly the same asset) and this asset cannot generate the required trade. Hence A-D equilibrium outcome would not be achieved. In general, we need at least two independent asses to complete the market with two states to guarantee the equivalence between A-D equilibrium and Radner equilibrium.

(e) Now there are two states and two independent assets, hence the A-D equilibrium outcome can be achieved in Radner equilibrium using those two assets.

**Answer for Q2**

(a) The derivative of the objective function at $\alpha = 0$ is $E[W(\widetilde{r}-1)\,u'(0)] = Wu'(0)E[(\widetilde{r}-1)] > 0$. hence $\alpha^* = 0$ is not the optimal solution.

(b) The first order condition in this case becomes

$$E[2(\widetilde{r}-1)(\alpha(\widetilde{r}-1)-1)] = 0.$$

Hence

$$\begin{aligned} & \alpha E[(\widetilde{r}-1)^2] - E[(\widetilde{r}-1)] \\ = \;& \alpha\left(VAR(\widetilde{r}) + (E[\widetilde{r}]-1)^2\right) - (E[\widetilde{r}]-1) \\ = \;& 0. \end{aligned}$$

Thus

$$\alpha^* = \frac{E[\widetilde{r}]-1}{VAR(\widetilde{r}) + (E[\widetilde{r}]-1)^2}.$$

(c) For (b), $W$ can be simply factored out of the objective function. More generally, $\alpha^*$ is independent of $W$ for CRRA utility function: $u(x) = \frac{x^{\gamma--1}}{\gamma-1}$, $\gamma \leq 1$. Since $u(\alpha W\widetilde{r} + (1-\alpha)W) = W^{\gamma-1}\frac{(\alpha\widetilde{r}+(1-\alpha))^{\gamma-1}}{\gamma-1}$, the size of wealth does not affect the optimal fraction $\alpha^*$.

(d) Since $\widetilde{r}$ SOSD $\widetilde{R}$, $\widetilde{R}$ can be expressed as $\widetilde{r}$ plus noise, i.e. $\widetilde{R} = \widetilde{r} + \varepsilon$, where $E[\varepsilon|\widetilde{r}] = 0$. This implies that $E[\widetilde{R}] = E[\widetilde{r}]$ and $VAR(\widetilde{R}) = VAR(\widetilde{r}) + VAR(\varepsilon) > VAR(\widetilde{r})$. Hence $\alpha^*$ would become smaller.

1. **Answer for Q3**

   (a) When types $\theta \in [0, \theta^*]$ send the signal $\varnothing$, the price $p = E[\theta|\varnothing] = \theta^*/2$ since the type distribution is uniform.

   (b) The threshold type $\theta = \theta^*$ gets price, $p = \theta^*$, by revealing and $p = \theta^*/2$ by withholding. In equilibrium he must be indifferent, so $\theta^* = 0$. In other words, all types reveal their type.

   (c) With probability $\alpha(1-\theta^*)$, the signal $s = \varnothing$ comes from types $\theta \geq \theta^*$ and has an expected value of $(1+\theta^*)/2$; with probability $\theta^*$, it comes from types $\theta \leq \theta^*$ and has an expected value of $\theta^*/2$. Its expected value is thus

   $$\frac{\alpha(1-\theta^*)(1+\theta^*)/2 + \theta^*\theta^*/2}{\alpha(1-\theta^*)+\theta^*}.$$

   In equilibrium type $\theta^*$ must be indifferent between the price he gets from the buyer receiving signals $\varnothing$ or $\theta^*$, that is

   $$\begin{aligned} \theta^*(\alpha(1-\theta^*)+\theta^*) &= \alpha(1-\theta^*)(1+\theta^*)/2 + \theta^*\theta^*/2 \\ \theta^*\theta^*/2 &= \alpha(1-\theta^*)(1-\theta^*)/2 \end{aligned}$$

   which is solved by $\theta^* = 1/4$. This threshold type achieves the same price when his type is revealed as when he pools with all lower types $\theta \leq \theta^*$, as well as with proportion $\alpha$ of higher tyes $\theta \geq \theta^*$.

   (d) In part b) the seller's utility equals his type $\theta$. In part c), low types $\theta \leq \theta^* = 1/4$ get utility 1/4, while high types $\theta \geq \theta^*$ get utility $\frac{8}{9}\theta + \frac{1}{9}\frac{1}{4}$. Thus, low types are better off (since they are pooling with some high types) while high types are worse off (since they are sometimes forced to pool with low types).

   (e) As $\alpha$ rises, the inference (and hence the price) from the $\varnothing$ signal improves, and so $\theta^*$ rises. Low types $\theta \leq \theta^*$ benefit while high types $\theta \geq \theta^*$ lose out.

1. **Answer for Q4**

   (a) Here a strategy is a mapping $s_i : \{1, 2, ..., M\} \rightarrow \{trade, not\ trade\}$. Any strategy for $i$ is rationalizable $\forall i$ since it is a best response to "never trade $\forall j \neq i$".

   (b) We first argue that in no equilibrium will $i$ use a strategy placing positive probability on trading when his card is $M$. This strategy can only be optimal if the next person to $i$'s right who trades with positive probability, agent $j$, trades if and only if his card reads $M$. Iterating this argument for agent $j$ and so on, the strategy profile would consist of a subset of agents $I \subseteq N$ who trade with positive probability if and only if their card reads $M$ and other agents $N \backslash I$ not trading at all. Clearly this is not an equilibrium since any agent would benefit from trading also when his card reads 1.

   Iterating this proof rules out placing positive probability on any number 2 through $M - 1$. Thus, the set of candidate strategies is reduced to "if card reads 1, trade with probability $p \in [0, 1]$; if card reads 2 to $M$, do not trade". Clearly this is a BNE for any $p$.

Question 5

(a) $S^*(\theta_1,\theta_2) = (\theta_1+\theta_2-k)q^*(\theta_1,\theta_2) = \underset{q}{Max}\{(\theta_1+\theta_2-k)q\}$

Then

$$(\theta_1+\theta_2-k)q^*(\theta_1,\theta_2) \geq (\theta_1+\theta_2-k)q^*(x_1,\theta_2) \text{ for all } \theta_2 \quad (1)$$

Note that the optimal allocation rule is

$$q^*(\theta) = \begin{cases} 0, & \theta_1+\theta_2 < k \\ 1, & \theta_1+\theta_2 \geq k \end{cases}.$$

**Net contribution mechanism:**

Under truth-telling, agent 1's payoff is

$$S^*(\theta_1,\theta_2) - S^*(\alpha,\theta_2) = (\theta_1+\theta_2-k)q^*(\theta_1,\theta_2) - (\alpha_1+\theta_2-k)q^*(\alpha_1,\theta_2) .$$

We can ignore the second term since it is independent of agent 1's report.

If agent 2 tells the truth, agent 1's payoff is therefore

$$(\theta_1+\theta_2-k)q^*(x_1,\theta_2) - (\alpha_1+\theta_2-k)q^*(\alpha_1,\theta_2).$$

By (1) truth telling is agent 1's best response.

(b) If the reports are $(x_1,x_2)$, then agent 1's payoff is

$$(\theta_1+x_2-k)q^*(x_1,x_2) - (\alpha_1+x_2-k)q^*(\alpha_1,x_2) \quad (2)$$

We can rewrite (1) as follows:

$$(\theta_1+x_2-k)q^*(\theta_1,x_2) \geq (\theta_1+x_2-k)q^*(x_1,x_2) \text{ for all } x_2 \quad (3)$$

Appealing to (3), agent 1's best response is still to report the truth.

(c)

Case (i) $S^*(\alpha,\theta_2) = \underset{q}{Max}\{(\alpha+\theta_2-k)q\} = 0$ , since $\alpha+\theta_2 < k$

Therefore under truth tell agent 1's payoff is $U_1 = S^*(\theta_1,\theta_2) - S^*(\alpha,\theta_2) = S^*(\theta_1,\theta_2)$ .

Same for agent 2.

But $U_D + U_1 + U_2 = S^*$ . Therefore $U_D = -S^*(\theta) < 0$ .

Case (ii). Same as case (i) for agent 1 so $U_1 = S^*$

For agent 2,

$$S^*(\alpha, \theta_2) = \alpha + \theta_2 - k \text{ since } \alpha + \theta_2 > k$$

$$S^*(\theta_1, \theta_2) = \theta_1 + \theta_2 - k .$$

Therefore

$$U_2 = S^*(\theta_1, \theta_2) - S^*(\theta_1, \alpha) = \theta_2 - \alpha > 0$$

Therefore $U_1 + U_2 > S^*$

(d) As in c (ii)

$$U_2 = S^*(\theta_1, \theta_2) - S^*(\theta_1, \alpha) = \theta_2 - \alpha$$

By the same argument,

$$U_1 = S^*(\theta_1, \theta_2) - S^*(\alpha_1, \theta_2) = \theta_1 - \alpha$$

Therefore

$$U_1 + U_2 = \theta_1 + \theta_2 - 2\alpha = (\theta_1 + \theta_2 - k) + (k - 2\alpha) = S^* + (k - 2\alpha) > S^* .$$

Thus the designer always has a negative payoff

(e) Example of a profitable mechanism.

Monopolist announces that the project will proceed if and only if each report is at least $(k + \pi)/2$ . Then reporting the truth is a best response.

The project goes forward if $\theta_j \geq (k + \pi)/2, \quad j = 1, 2$ and the designer profit is $\pi$ in that eventuality.

**Answer to question 6**

Since $\{q(\theta), r(\theta)\}_{\theta \in X}$ is incentive compatible,

$$U(\theta) = \theta q(\theta) - r(\theta) = \underset{z}{Max}\{\theta q(z) - r(z)\} \ .$$

Appealing to the Envelope Theorem, $U'(\theta) = q(\theta)$

$$\mathrm{E}[U(\theta)] = \int_0^1 U(\theta) f(\theta) d\theta$$

$$= U(\theta)(F(\theta)-1)\Big|_0^1 - \int_0^1 U'(\theta)(F(\theta)-1) d\theta$$

$$= U(0) + \int_0^1 U'(\theta)(1-F(\theta)) d\theta$$

$$= U(0) + \int_0^1 q(\theta)(\frac{1-F(\theta)}{f(\theta)}) f(\theta) d\theta \ .$$

Also

$$\mathrm{E}[U(\theta)] = \mathrm{E}[\theta q(\theta) - r(\theta)]$$

Therefore

$$\mathrm{E}[r(\theta)] = \mathrm{E}[\theta q(\theta)] - \mathrm{E}[U(\theta)] = \int_0^1 q(\theta)(\theta - \frac{1-F(\theta)}{f(\theta)}) f(\theta) d\theta - U(0)$$

$$= \int_0^1 q(\theta) J(\theta) f(\theta) d\theta - U(0) \ .$$

$$= \mathrm{E}[q(\theta) J(\theta)] - U(0)$$

$$J_A(\theta) = \theta - \frac{(1-F_A(\theta)}{f_A(\theta)} = \theta - \frac{1-\theta}{1} = 2\theta - 1$$

(b) If the seller sets a price of $p$ the allocation is $q(\theta) = 1$ if and only if $\theta \geq p$ . So choose $p^*$ to satisfy $J_A(p^*) = 0$ . So $p^* = 1/2$

(c) $J_A(\theta)$ is strictly increasing. Thus point-wise maximization of the integrand is achieved by

$$q(\theta)=\begin{cases}0, & \theta < p^{*} \\ 1, & \theta \geq p^{*}\end{cases}.$$

(d) $J_B(v)=\theta-\dfrac{1-F_B(v)}{f_B(v)}=\theta-\dfrac{1-2\theta}{2}=2\theta-\tfrac{1}{2}$

Values are independent so treat the items separately. Then set a price of ¼ for the second item.

(e) By independence, auction off items A and B separately

For item A, let $q_1(\theta)$, $q_2(\theta)$ be the allocation rule

$$\mathrm{E}[r_1(\theta)]+\mathrm{E}[r_2(\theta)]=\mathrm{E}[q_1(\theta)J_A(\theta_1)+q_2(\theta)J_A(\theta_2)]$$

Allocate to the person with the highest positive $J_A$ so set a reserve price of ½ for the first item and use a standard auction.

Similarly auction item B with a reserve price of ¼ .

Use either of the common auctions.