Comprehensive Examination
Quantitative Methods
Spring, 2017

Instruction: This exam consists of three parts. You are required to answer all the questions in all the parts.

Grading policy:

1. Each part will be graded separately, and there are four possible results in each part: H (honor pass), P (PhD pass), M (master pass), and F (fail).
2. Each part contains a precise grade determining algorithm.
3. The grades from the three parts will be summarized in the descending order, after which the overall grade will be determined using the algorithm summarized in the table below:

Highest	Middle	Lowest	**Overall**
H	H	H	**H**
H	H	P	**H**
H	H	M	**P**
H	H	F	**M**
H	P	P	**P**
H	P	M	**P**
H	P	F	**M**
H	M	M	**M**
H	M	F	**M**
H	F	F	**F**
P	P	P	**P**
P	P	M	**P**
P	P	F	**M**
P	M	M	**M**
P	M	F	**M**
P	F	F	**F**
M	M	M	**M**
M	M	F	**F**
M	F	F	**F**
F	F	F	**F**

Part I - 203A

Instruction for Part I: Solve every question.
Grading policy for Part I: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let T denote the total number of points.

1. If $T \geq 25$, you will get H.
2. If $20 \leq T < 25$, you will get P.
3. If $15 \leq T < 20$, you will get M.
4. If $T < 15$, you will get F.

Question 1 (10 points): The time, Y, it takes a worker to perform a given task is distributed $\exp(X)$, where X is a measure of his/her experience. Hence, the conditional density of Y given X is:

$$f_{Y|X=x}(y) = \begin{cases} x\ e^{-xy} & if \qquad y > 0 \\ 0 & otherwise \end{cases}$$

There are two types of workers, 1 and 2. Let Z denote the type of worker and p denote the probability that a worker is of type 1. The conditional density of X given Z is:

$$f_{X|Z=1}(x) = \begin{cases} \frac{1}{a_2 - a_1} & if \qquad a_1 < x < a_2 \\ 0 & otherwise \end{cases}$$

$$f_{X|Z=2}(x) = \begin{cases} \frac{1}{b_2 - b_1} & if \qquad b_1 < x < b_2 \\ 0 & otherwise \end{cases}$$

where $0 < a_1 < b_1 < a_2 < b_2$, and where Z only affects X.

a. (8 points) Obtain the simplest possible expressions in terms of the parameters a_1, a_2, b_1, and b_2 for

a.1. (2 points) The probability that a worker is of type 1 given that his/her experience is known to be larger than b_1.

a.2. (3 points) The expected time that it will take a worker to perform the task if his/her experience is known to be larger than b_1.

a.3. (3 points) The probability that a worker is of type 1 given that the time it took him/her to perform the task was observed to be larger than $\overline{y}$.

b. (2 points) Provide answers and justifications.

b.1. (1 point) Are Y and Z independent?

b.2. (1 point) Are Y and Z independent, conditional on X?

Question 2 (20 points): The value of a random variable Y^* is determined by X_1, X_2 and ε according to the relationship

$$Y^* = m\left(\beta_1' X_1, \beta_2' X_2\right) - \varepsilon$$

where $m : R^2 \to R$ is a continuously differentiable but otherwise unknown function, (X_1, X_2) is a continuously distributed random vector with support $R^{K_1+K_2}$, where $X_1 \in R^{K_1}$ and $X_2 \in R^{K_2}$, β_1 and β_2 are parameters of unknown value, in respectively R^{K_1} and R^{K_2}, and ε is a random variable distributed independently of (X_1, X_2) with a strictly increasing, differentiable but otherwise unknown cumulative distribution F_ε.

Suppose that only (Y, X_1, X_2) is observable, where

$$Y = \begin{cases} 1 & if \quad Y^* > 0 \\ 0 & otherwise \end{cases}$$

a. (5 points) Is the vector of parameters (β_1, β_2) identified? If your answer is YES, prove it. If your answer is NO, provide a counterexample, specify a set of *minimal additional conditions on* (β_1, β_2) under which (β_1, β_2) is identified, and show that under those additional conditions, (β_1, β_2) is identified.

b. (5 points) Assume (β_1, β_2) is known. Is the distribution, F_ε, of ε identified? If your answer is YES, prove it. If your answer is NO, provide a counterexample, specify a set of *additional, nonparametric conditions on* m under which F_ε is identified, and show that under those additional conditions, F_ε is identified.

c. (10 points) Suppose now that $K_1 = K_2 = 2$. Assume that (β_1, β_2) is known, the function m is known and is strictly increasing in each of its arguments, and the distribution F_ε is also known. Obtain an expression for the density of Y^* in terms of m, F_ε, β_1, β_2, and the distribution of the observable vector (Y, X_1, X_2). You may make additional assumptions if you justify their need.

Part II - 203B

Instruction for Part II: Solve every question.
Grading policy for Part II: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality. Let T denote the total number of points.

1. If $T \geq 25$, you will get H.
2. If $20 \leq T < 25$, you will get P.
3. If $15 \leq T < 20$, you will get M.
4. If $T < 15$, you will get F.

Question 1 (5 pts.) *No derivation is needed for this question; your derivation will not be read anyway.* Suppose that

$$E\left[x_i\left(y_i - x_i\theta\right)\right] = 0$$
$$E\left[z_i\left(y_i - x_i\theta\right)\right] = 0$$

Let $\widehat{\theta}$ denote the optimal (two step) GMM estimator based on the two moments above. Suppose that $(y_i, x_i, z_i)'$ $i = 1, 2, \ldots$ are i.i.d. Under standard regularity conditions, $\sqrt{n}\left(\widehat{\theta} - \theta\right)$ converges in distribution to $N(0, \Omega)$ for some Ω. Assume further that

$$\begin{bmatrix} x_i \\ z_i \\ \varepsilon_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1.2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3.6 \end{bmatrix}\right),$$

where $\varepsilon_i = y_i - x_i\theta$. What is the numerical value of Ω?

Question 2 (5 pts.) *No derivation is needed for this question; your derivation will not be read anyway.* Suppose that $x_i, y_i^*, \varepsilon_i$ are scalars such that

$$y_i^* = \beta_1 + x_i\beta_2 + u_i.$$

We observe (y_i, D_i, x_i) $i = 1, \ldots n$, where

$$D_i = 1\left(u_i > 0\right)$$
$$y_i = D_i \cdot y_i^*$$

We assume that u_i is independent of x_i, and

$$u_i \sim N(0, 4)$$

As usual, we assume that (y_i, D_i, x_i) $i = 1, \ldots n$ are i.i.d.

Suppose that we regress y_i on $(1, x_i)'$ in the subsample where $D_i = 1$. Let $\left(\widehat{\beta}_1, \widehat{\beta}_2\right)'$ denote such an OLS estimator. Let

$$(\alpha_1, \alpha_2) = \text{plim}\left(\widehat{\beta}_1 - \beta_1, \widehat{\beta}_2 - \beta_2\right).$$

Provide a numerical characterization of (α_1, α_2). Your answer must take the form of *a concrete numerical value of a two-dimensional vector.* The only symbols allowed in your answer are π and e. Any other abstract formula, even the one involving $\lambda(\cdot)$, will be given zero credit.

Question 3 (4 pts.) *No derivation is needed for this question; your derivation will not be read anyway.* Suppose that $X = (X_1, \ldots, X_{16})'$, where X_i are i.i.d. $N(\theta_1, \theta_2)$. Assuming that $(\theta_1, \theta_2) = (1, 4)$. What is the numerical value of the Fisher Information for θ based on X? Do not forget that there are 16 components of X.

Question 4 (5 pts.) *No derivation is needed for this question; your derivation will not be read anyway.* Suppose that

$$\begin{aligned} y_1 &= \mu + \varepsilon_1 \\ y_2 &= 3\mu + \varepsilon_2 \\ y_3 &= 5\mu + \varepsilon_3 \end{aligned}$$

such that for $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)'$, we have

$$E[\varepsilon] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad E[\varepsilon\varepsilon'] = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & 9\sigma^2 & 0 \\ 0 & 0 & 25\sigma^2 \end{bmatrix}$$

for some $\sigma^2 > 0$. We observe $y = (y_1, y_2, y_3)'$. What is the best linear unbiased estimator of μ if $y = (3, 6, 5)'$? Your answer should be a number.

Question 5 (5 pts.) *No derivation is needed for this question; your derivation will not be read anyway.* Let

$$\begin{aligned} y^{(j)} &= X\beta + \varepsilon^{(j)}, \\ \widehat{\beta}^{(j)} &= \left(X'X\right)^{-1} X' y^{(j)}, \\ D_{(j)} &= \begin{cases} 1 & \text{if } \widehat{\beta}^{(j)} - \frac{z_{0.01}}{\sqrt{30}} \leq 1 \leq \widehat{\beta}^{(j)} + \frac{z_{0.01}}{\sqrt{30}} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $\varepsilon^{(1)}, \ldots, \varepsilon^{(M)}$ are independent $N(0, I_4)$ random vectors,

$$X = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \text{and} \quad \beta = 1.$$

(Here, z_α is a number such that $\Pr(Z \geq z_\alpha) = \alpha$ for $Z \sim N(0, 1)$.) What is $\text{plim}_{M\to\infty} \frac{1}{M} \sum_{j=1}^{M} D_{(j)}$? Your answer should be a number.

Question 6 (4 pts.) *No derivation is needed for this question; your derivation will not be read anyway.* Suppose that

$$y_i = x_i\beta + z_i\gamma + \varepsilon_i$$

such that (i) $(x_i, z_i, \varepsilon_i)$ $i = 1, 2, \ldots, n$ are i.i.d.; (ii) $E[x_i] = E[z_i] = E[\varepsilon_i] = 0$; (iii) $E[x_i\varepsilon_i] = E[z_i\varepsilon_i] = E[x_iz_i] = 0$. Let $\widetilde{\beta}$ denote the OLS coefficient when y_i is regressed on x_i. Let $\left(\widehat{\beta}, \widehat{\gamma}\right)$ denote the OLS estimator when y_i is regressed on (x_i, z_i). Your task is to characterize the asymptotic distributions of $\sqrt{n}\left(\widetilde{\beta} - \beta_*\right)$ and $\sqrt{n}\left(\widehat{\beta} - \beta_*\right)$. In your answer below, assume that

$$E\left[x_i^2\right] = 2, \quad E\left[z_i^2\right] = 2.5, \quad E\left[\varepsilon_i^2\right] = 3, \quad \beta = 1, \quad \gamma = 4.$$

(a) (2 pts.) What is the asymptotic distribution of $\sqrt{n}\left(\widetilde{\beta} - \beta_*\right)$? Your answer should be numerical. You will get zero credit otherwise.

(b) (2 pts.) What is the asymptotic distribution of $\sqrt{n}\left(\widehat{\beta} - \beta_*\right)$? Your answer should be numerical. You will get zero credit otherwise.

Question 7 (2 pts.) *No derivation is needed for this question; your derivation will not be read anyway.* Suppose that X is a $n \times k$ matrix with full column rank, and that $y = X\beta + \varepsilon$, where $\varepsilon \sim N(0, I_n)$. Let $\widehat{\beta} = (X'X)^{-1} X'y$, and $e = y - X\widehat{\beta}$.

(a) (1 pt.) What is $\mathrm{Var}(e'e)$? You should express it as an explicit function of n, and k. A loose descriptive answer will be given zero credit.

(b) (1 pt.) Assume that $k = k_n$ changes as $n \to \infty$ in such a way that $\lim_{n\to\infty} k_n = \infty$ but $\lim_{n\to\infty} \frac{k_n}{n} = \frac{1}{2}$. What is $\operatorname{plim} \frac{e'e}{n}$? Your answer should be a number.

Part III - 203C

Instruction for Part III: Solve every question.
Grading policy for Part III: Your grade in this part of the exam is based on the total points that you earn. Below is how the grade is determined. In order to avoid any confusion arising from the difference between strict and weak inequalities, the grade assignment algorithm is presented in terms of mathematical inequalities. Note that $<$ denotes a strict inequality, and $\leq$ denotes a weak inequality.

Let T denote the total number of points.

1. If $T \geq 25$, you will get H.
2. If $20 \leq T < 25$, you will get P.
3. If $15 \leq T < 20$, you will get M.
4. If $T < 15$, you will get F.

Question 1 (9 points) Let $\{u_t\}_{t\in\mathbb{Z}}$ be an i.i.d. process of standard normal random variables.

(a) (2 points) Define $X_t = u_t + u_{t-1}$ for any $t \in \mathbb{Z}$. Find the mean and auto-covariance function of $\{X_t\}_{t\in\mathbb{Z}}$.

(b) (2 points) Define $Y_t = u_t u_{t-1}$ for any $t \in \mathbb{Z}$. Find the mean and auto-covariance function of $\{Y_t\}_{t\in\mathbb{Z}}$.

(c) (2 points) For any $t \in \mathbb{Z}$,

$$W_t = \begin{cases} u_t, & t \text{ even} \\ 2^{-1/2}(u_t^2 - 1), & t \text{ odd} \end{cases}.$$

Is $\{W_t\}_{t\in\mathbb{Z}}$ covariance stationary? Justify your answer.

(d) (3 points) Is $\{W_t\}_{t\in\mathbb{Z}}$ defined in (c) strictly stationary? Justify your answer.

Question 2 (6 points) Let $\{X_t\}_{t\in\mathbb{Z}}$ be a time series generated by

$$X_t = e_1 \cos(t) + e_2 \sin(t) + Z_t,$$

where $(e_1, e_2)'$ is distributed as a bivariate normal random variable with zero mean and covariance matrix

$$\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

$Z_t = u_t - u_{t-1}$, and $\{u_t\}_{t\in\mathbb{Z}}$ is an $IID(0, 3)$ process, independent of $(e_1, e_2)'$.

(a) (3 points) Find the auto-covariance function of $\{X_t\}_{t\in\mathbb{Z}}$.

(b) (3 points) Find the spectral density of $\{X_t\}_{t\in\mathbb{Z}}$.

Question 3 (15 points) Suppose that $\{X_t\}_{t\in\mathbb{Z}}$ is a stationary time series which satisfies

$$X_t - \mu = \phi(X_{t-1} - \mu) + u_t \tag{1}$$

where $|\phi| < 1$, μ is a finite constant, $\{u_t\}_{t\in\mathbb{Z}} \sim IID(0, \sigma^2)$ and $E[u_t^4] < \infty$. Suppose we have data $\{X_t\}_{t=1}^n$.

(a) (4 points) Let $\overline{X}_n = n^{-1}\sum_{t=1}^n X_t$. Derive the asymptotic distribution of $\overline{X}_n$.

(b) (2 points) In a sample of size 100 from the process $\{X_t\}_{t\in\mathbb{Z}}$ with $\phi = 0.6$ and $\sigma^2 = 10$, we obtain $\overline{X}_n = 0.271$. Construct an approximate 95% confidence interval for the mean μ. Does the data suggest that $\mu = 0$?

(c) (4 points) Define

$$\widehat{\phi}_{n,1} = \frac{\sum_{t=1}^n X_t X_{t-1}}{\sum_{t=1}^n X_{t-1}^2}.$$

Is $\widehat{\phi}_{n,1}$ a consistent estimator of ϕ? Justify your answer.

(d) (5 points) Define

$$\widehat{\phi}_{n,2} = \frac{\sum_{t=1}^n (X_t - \overline{X}_n)(X_{t-1} - \overline{X}_n)}{\sum_{t=1}^n (X_{t-1} - \overline{X}_n)^2}.$$

Is $\widehat{\phi}_{n,2}$ a consistent estimator of ϕ? Justify your answer.

A Useful Theorem

Theorem 1 *Suppose that X is a normal random variable with mean μ and variance σ^2. Then*

$$E[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4.$$

0Moreover,

$$\Pr\left(\frac{X-\mu}{\sigma} \leq 0\right) = 0.5 \text{ and } \Pr\left(\frac{X-\mu}{\sigma} \leq 1\right) = 0.8413.$$